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Intermediate action for the four-components link

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Marcelo Antonio Anda Chavarría

Director de tesis: Ernesto Contreras, Ph.D.

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Marcelo Antonio Anda Chavarría

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Nombres y apellidos:	Marcelo Antonio Anda Chavarría
Código:	00329960
Cédula de Identidad:	0956887939
Lugar y fecha:	Quito, 15 de Diciembre de 2023

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"And here, I am afraid I must end by saying that the difficulties are so great in the way of forming anything like a comprehensive theory that we cannot even imagine a finger-post pointing to a way that lead us towards the explanation. That is not putting it too strongly. I can only say we cannot now imagine it. But this time next year,– this time ten years, - this time one hundred years, – probably it will be just as easy as we think it is to understand that glass of water, which seems now so plain and simple. I cannot doubt but that these things, which now seem to us so mysterious, will be no mysteries at all; that the scales will fall from our eyes; that we shall learn to look on things in a different way– when that which is now a difficulty will be the only common-sense and intelligible way of looking at the subject."

- William Thomson,

later known as Lord Kelvin, in his 1889 Presidential Address to the Institution of Electrical Engineers on his failed vortex theory of the atom [1].

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RESUMEN

En 2002, Lorenzo Leal desarrolló las ecuaciones clásicas de movimiento para la teoría de Chern-Simons acoplada con partículas de Wong [2]. En el contexto de la teoría de nudos, es evidente que al definir formas específicas para las corrientes de las partículas en términos de caminos cerrados en el espacio-tiempo, la acción "on-shell" se relaciona directamente con un invariante de nudo. Sin embargo, la presencia de un sistema de ecuaciones no lineales acopladas dificulta la derivación de una solución analítica. Al emplear un método perturbativo, se tiene que a cada orden le corresponde un invariante de nudo distinto. El término a orden cero corresponde al número de anudamiento de Gauss, mientras que el término a primer orden determina el anudamiento de los anillos Borromeanos. Para aliviar la complejidad del análisis perturbativo, se ha demostrado que estos términos también se pueden derivar partiendo desde una teoría de campos abeliana acoplada con las corrientes apropiadas [3]. Este trabajo adopta una "metodología de acción intermedia" para construir una expresión analítica para el enlace de cuatro componentes.

Palabras clave: teoría de Chern-Simons, teoría de nudos, invariante de nudo, invariante de enlace, número de anudamiento de Gauss, anillos Borromeanos, teoría topológica de campos, enlace de cuatro componentes, partículas de Wong.

ABSTRACT

In 2002, Lorenzo Leal developed the classical equations of motion within the Chern-Simons theory coupled with Wong Particles [2]. In the realm of knot theory, it is apparent that by defining specific forms of currents in terms of closed loops in space-time, the "on-shell" action relates directly to a knot invariant. Yet, the presence of a system of coupled non-linear equations hinders an analytical solution derivation. Employing perturbative methods reveals that each order corresponds to a distinct knot invariant. The zeroth-order term corresponds to the Gauss Linking number, while the first-order term quantifies the linking of Borromean rings. To alleviate the potential complexity of perturbation, it has been demonstrated that these terms can also be derived by initiating from an Abelian field theory coupled with appropriate currents [3]. This work adopts such an "intermediate action methodology" to construct an expression for the four-components link.

Keywords: Chern-Simons theory, knot theory, knot invariant, link invariant, Gauss linking number, Borromean rings, topological field theory, four-components link, Wong particles.

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Introduction

Chern-Simons theory is a topological quantum field theory, meaning that physical observables are independent of the metric. In the late 1980s, Edward Witten demonstrated that the expected values of non-local observables (Wilson loops) in Chern-Simons theory, which can be represented by knots in three-dimensional space, correspond to knot invariants generalizing Jones polynomials [4]. Since then, there have been significant advances in the study of the theory from both a physical and mathematical perspective.

In 2002, a study was conducted by considering the classical equations of motion of Chern-Simons theory coupled with particles carrying chromoelectric charge (non-Abelian Wong particles) [2]. The action of the Chern-Simons-Wong theory is given by:

$$S = S_{CS} + S_{int}, \tag{1}$$

$$S_{CS} = -\Lambda^{-1} \int d^3x \varepsilon^{\mu\nu\rho} \, \mathrm{Tr} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right), \tag{2}$$

$$S_{int} = \sum_{i=1}^{n} \int_{\gamma_i} d\tau \, \mathrm{Tr} \left(K_i g_i^{-1}(\tau) D_\tau g_i(\tau) \right), \tag{3}$$

where S_{CS} is the Chern-Simons action for $SU(N)$, S_{int} corresponds to the field-particle interaction of n Wong particles, Λ is a constant, $\varepsilon^{\mu\nu\rho}$ is the levi-civita symbol, A_μ is the electromagnetic potential, $g_i(\tau)$ are matrices associated with the internal degrees of freedom of the particles, $D_\tau g_i(\tau)$ is the covariant derivative of $g_i(\tau)$ along the worldline of the i^{th} particle, and K_i is a constant element of the algebra related to the initial value of the chromo-electric charge $I_i(\tau)$ defined by:

$$I_i(\tau) \equiv g_i(\tau) K_i g_i^{-1}(\tau). \tag{4}$$

In the context of knot theory, it is observed that for a particular form of the current in terms of closed paths in space-time, the "on-shell" action corresponds to a link invariant. However, because a system of coupled non-linear equations is obtained, it is not possible to find analytical solutions. Therefore, a perturbative solution method was proposed [2]. In that work, the first two terms of the perturbative series were found. The zeroth-order term corresponds to the second Milnor's linking coefficient:

$$S^{(0)} = \frac{1}{4}\sum_{i,j} I_i^a(0) I_j^a(0) L(i,j), \tag{5}$$

where

$$L(i,j) \equiv \frac{1}{4\pi}\oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\rho \frac{(z-y)^\beta}{|z-y|^3}\epsilon_{\mu\nu\rho}, \tag{6}$$

corresponds to the Gauss linking number (GLN) of γ_i and γ_j. The first-order term of the perturbative series is the third Milnor's coefficient (TMC), which measures the linking of Borromean rings and is given by:

$$\begin{aligned} S^{(1)}(1,2,3) = &-\frac{1}{2}\int d^3x \epsilon^{\mu\nu\rho} D_{1\mu}(x) D_{2\nu}(x) D_{3\rho}(x) - \\ &-\frac{1}{2}\int d^3x \int d^3y \left\{ T_1^{[\mu x,\nu y]} D_{2\mu}(x) D_{3\nu}(y) + \right. \\ &+ T_2^{[\mu x,\nu y]} D_{3\mu}(x) D_{1\nu}(y) + \\ &\left. + T_3^{[\mu x,\nu y]} D_{1\mu}(x) D_{2\nu}(y) \right\}, \end{aligned} \tag{7}$$

where the bilocal object $T_{\gamma_i}^{\mu x,\nu y}$ associated with the curve γ_i is introduced:

$$T_{\gamma_i}^{\mu x,\nu y} \equiv \oint_{\gamma_i} dz^\mu \int_0^z dz'^\nu \delta^3(x-z)\delta^3(y-z'), \tag{8}$$

and we have used the definitions:

$$T_{\gamma_i}^{[\mu x,\nu y]} \equiv \frac{1}{2}\left(T_{\gamma_i}^{\mu x,\nu y} - T_{\gamma_i}^{\nu x,\mu y}\right), \tag{9}$$

$$D_{i\alpha}(x) \equiv \frac{1}{4\pi}\oint_{\gamma_i} dz^\gamma \frac{(x-z)^\beta}{|x-z|^3}\epsilon_{\alpha\beta\gamma}. \tag{10}$$

On the other hand, the Abelian Chern-Simons theory only reproduces the GLN. For this reason, the question about if there exists an Abelian theory that can exactly reproduce, for example, the TMC naturally arises. Thus, it seems feasible to study topological theories intermediate between Abelian and non-Abelian Chern-Simons theories. In [3], an intermediate Abelian action is proposed, which exactly reproduces the first term of the non-Abelian perturbative series of Chern-Simons-Wong theory, and is given by

$$\begin{aligned} S = &\int d^3x \varepsilon^{\mu\nu\rho}\left\{ 4A_\mu^i(x)\partial_\nu a_{i\rho}(x) + \frac{2}{3}\varepsilon^{ijk} a_{i\mu}(x) a_{j\nu}(x) a_{k\rho}(x)\right\} - 2\int d^3x T_i^{\mu x} A_\mu^i(x) + \\ &+ \int d^3x \int d^3y \varepsilon^{ijk} T_i^{\mu x,\nu y} a_{j\mu}(x) a_{k\nu}(y), \end{aligned} \tag{11}$$

where $A_u^i(x)$ and $a_u^i(x)$ are two independent sets of Abelian gauge fields. In this work, we propose to determine an Abelian intermediate action that reproduces the link invariant for the four-components link, similar to the non-Abelian Chern-Simons-Wong theory [5].

Chapter 1

Abelian Chern-Simons Theory

The Abelian Chern-Simons action is given by:

$$S_{\mathrm{CS}}[A] \equiv \frac{k}{4\pi} \int_M \mathrm{d}^3 x \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{1.1}$$

where M is the $(2+1)$-dimensional Minkowski space, $(\mu, \nu, \rho) \in \{0, 1, 2\}$, $A_\mu(x)$ is a $U(1)$ gauge field, and $k \in \mathbb{R}$ is called the *Chern - Simons level.* We will show in the next section that A_μ is an analogous to the photon field of electrodynamics. We will further investigate the properties of this theory when coupled with a matter field, particularly under a double exchange of particles to define an anyon. Additionally, we will explore its relationship with the GLN, highlighting its topological invariance. This chapter is based on the work of D. Grabovsky [6].

1.1 Equations of motion for the classical theory

Note that if we make variations of (1.1) with respect to the field A_μ, we obtain:

$$\delta S_{\mathrm{CS}} = 0 \;\Rightarrow\; \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} F_{\nu\rho} = 0 \;\Rightarrow\; F = 0, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.2}$$

trivial equations of motion, because there are no propagating local degrees of freedom. Note that the object $F_{\mu\nu}$ is formally equal to the Maxwell's field strength tensor for electromagnetism. In order to get non-trivial equations of motion we can couple the Chern-Simons action with a matter term, e.g., a Dirac fermion ψ which produces a current:

$$S = S_{CS} + S_\psi + S_{int}, \tag{1.3}$$

$$S_\psi = \int_M d^3x \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{1.4}$$

$$S_{int} = -\int_M d^3x e A_\mu \bar{\psi}\gamma^\mu \psi = \int_M d^3x A_\mu J^\mu, \tag{1.5}$$

where S_ψ is the Dirac action, S_{int} is an interaction term, ψ is a spinor, $\bar{\psi}(x) = \psi(x)^\dagger \gamma^0$ is the Dirac adjoint, $\partial\!\!\!/ = \gamma^\mu \partial_\mu$, and γ^μ are the Dirac gamma matrices. Again, if we consider variations of the action (1.3) with respect to the gauge field A_μ, we can write the equations of motion in terms of the current J^μ as follows:

$$\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}F_{\nu\rho} = \frac{k}{2\pi}\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho = J^\mu. \tag{1.6}$$

1.2 Electric and magnetic fields

The field strength tensor $F_{\mu\nu}$ in a $(2+1)$-dimensional Minkowski space is given by:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & B \\ -E_2 & -B & 0 \end{pmatrix}, \tag{1.7}$$

which is related with the electric E_i and magnetic B field as:

$$\begin{aligned} E_i &\equiv F_{0i} = -F_{i0} = -\partial_i A_0 + \partial_0 A_i, \\ B &\equiv \frac{1}{2}\varepsilon^{ij}F_{ij} = \varepsilon^{ij}\partial_i A_j, \end{aligned} \tag{1.8}$$

similar to Maxwell's theory in two dimensions. If we write the current in the form $J^\mu = (\rho, \vec{\mathrm{J}})$, it is easy to check that the temporal and spatial components of the equations of motion (1.6) are:

$$\rho = \frac{k}{2\pi}B, \quad J^i = \frac{k}{2\pi}\varepsilon^{ij}E_j, \tag{1.9}$$

respectively. Note that in this theory B is produced by electric charges, and $\vec{\mathrm{E}}$ is a consequence of the electric currents. Thus, the sources of the electric and magnetic fields in Chern-Simons theory have changed with respect to the electromagnetism. The physical situation is shown in Figure 1.1.

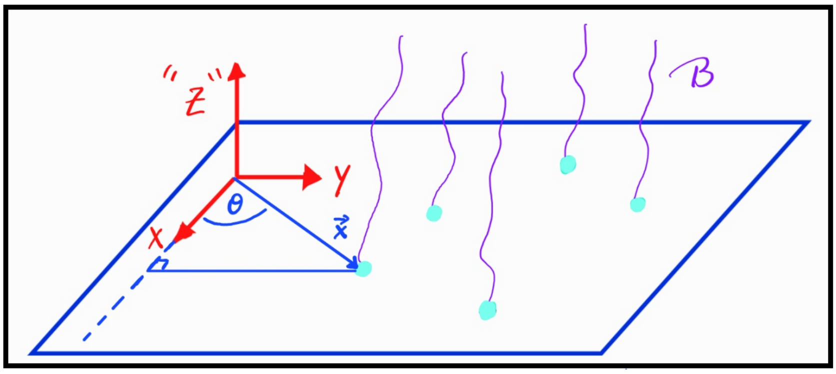

Figure 1.1: The moving charges over the plane generate the electric field $\vec{\mathrm{E}}$, while the magnetic field B for each particle points in an imaginary "z" direction. Actually, this scenario is physically accurate in the quantum Hall effect: each source charge generates magnetic flux lines that pass through the spatial manifold.

1.3 Double exchange of particles and anyons

Now, let us solve the CS equations (1.9) with sources $\rho = \delta^{(2)}\left(\vec{x} - \vec{x}_a(t)\right)$ and $\vec{\mathrm{J}} = \vec{0}$. We will work in Coulomb gauge, where $\vec{\nabla} \cdot \vec{\mathrm{A}} = \partial_i A^i = 0$. If we replace these sources into (1.9), write out the $\vec{\mathrm{E}}$ and B fields in terms of $A_\mu = (0, A_i)$ as in (1.8), and use the identity $\nabla^2 \log|\vec{x}| = 2\pi\delta^{(2)}(\vec{x})$, then the solution to (1.9) is given by

$$A_i(\vec{x}, t) = \frac{1}{k}\varepsilon_{ij}\frac{x^j - x_a^j(t)}{|\vec{x} - \vec{x}_a|^2} = -\frac{1}{k}\partial_i\theta\left(\vec{x} - \vec{x}_a(t)\right), \quad \theta(\vec{x}) \equiv \tan^{-1}\left(\frac{y}{x}\right) = \arg(\vec{x}). \tag{1.10}$$

Note that A_i is a total derivative, representing a pure gauge configuration that can vanish through a gauge transformation adding the total derivative of $\omega(x) \equiv \left(\frac{1}{k}\right)\theta\left(\vec{x} - \vec{x}_a(t)\right)$. Consequently, $A(x) \equiv 0$, wich implies $\vec{\mathrm{E}} = \vec{0}$ and $B = 0$. However, this gauge transformation also influences the matter field ψ, which acted as the source charge by imparting a nontrivial Aharanov-Bohm phase dependent on its angular position θ and on k,

$$\omega = \frac{1}{k}\theta \Longrightarrow \begin{cases} A_i(x) \longrightarrow A'(x) = A_i(x) + \partial_i\omega(x) = 0 \\ \psi(x) \longrightarrow \psi'(x) = e^{i\omega(x)}\psi(x) = e^{i\theta/k}\psi(x) \end{cases}. \tag{1.11}$$

Now, consider two charges where one remains stationary, while the other orbits the first as shown in Figure 1.2. We conceptualize this process as a double exchange, where the moving charge swaps positions with the stationary one twice, occurring once for each π rotation. According to equation (1.11), the phase acquired by the moving particle is expressed as

$$\Delta\theta = 2\pi \Longrightarrow \psi(x) \longrightarrow \exp\left(\frac{2\pi i}{k}\right)\psi(x) = \exp\left(i\oint_{\theta=0}^{\theta=2\pi} \mathrm{d}x^i A_i\right)\psi(x) \neq \psi(x). \tag{1.12}$$

If these particles were bosons or fermions, ψ would return to itself under double exchange. However, as indicated by (1.12), by tuning k, we can assign ψ arbitrary statistics valued in $U(1)$. Consequently, these particles are anyons, capable of exhibiting any statistics.

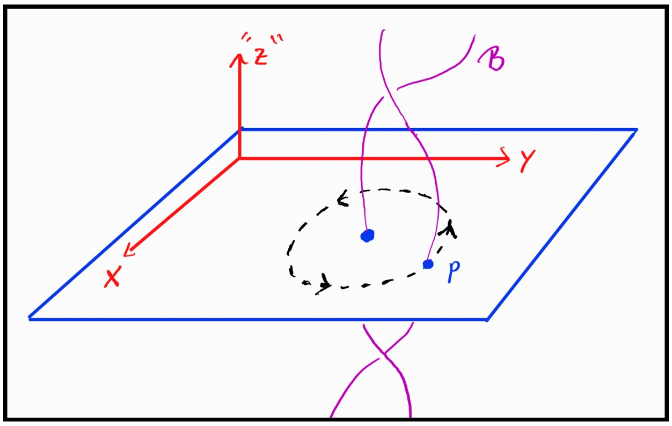

Figure 1.2: Anyons.

1.4 Gauss linking number

In this section, we will explore the topological invariance of the Abelian Chern-Simons theory and how a suitable current can lead us to obtain a link invariant. Recall that the sourced Chern-Simons action is given by

$$\tilde{S}_{CS} = S_{CS} + S_{int} = \int_M \mathrm{d}^3x \left(\frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu \right). \tag{1.13}$$

It might seem that the interaction term breaks topological invariance, but consider a "point-like" current density for two particles defined by

$$J^\mu = \sum_{a=1}^{2} J_a^\mu \equiv \sum_{a=1}^{2} \oint_{\gamma_a} \mathrm{d}x_a^\mu \delta^{(3)} \left(x - x_a(t)\right), \quad a \in \{1, 2\}. \tag{1.14}$$

This current accommodates topological invariance because it transforms like a vector density. Since the equations of motion (1.6) take the form of Ampere's Law, then its solutions have to be expressed as the Biot-Savart Law. Thus, the classical solution of $A_\mu(x)$ in terms of the current density (1.14) is given by

$$A_\mu(x) = \frac{1}{2k} \int_M \mathrm{d}^3y \varepsilon_{\mu\nu\rho} \frac{\partial^\nu J^\rho(y)}{|x-y|} = \frac{1}{2k} \sum_{a=1}^{2} \oint_{\gamma_a} \mathrm{d}x_a^\nu \varepsilon_{\mu\nu\rho} \frac{(x-x_a)^\rho}{|x-x_a|^3}. \tag{1.15}$$

Next, we substitute the solution (1.15) into $\tilde{S}_{CS}$ (1.13) in order to obtain $\tilde{S}_{CS\ on-shell}$. Since it is quadratic in A_μ, the sourced action includes terms involving two integrals over the loops γ_a. This characteristic captures the non-local aspect of interactions between the two localized particles. However, the terms that involve double integration over the same loop are divergent, depicting self-interactions. These terms persist even if J describes only a single particle. Thus, the on-shell action is given by

$$\tilde{S}_{CS\ on-shell} = \frac{1}{2k} \sum_{a,b} \oint_{\gamma_a} \mathrm{d}x_a^\mu \oint_{\gamma_b} \mathrm{d}x_b^\nu \varepsilon_{\mu\nu\rho} \frac{(x_1 - x_2)^\rho}{|x_1 - x_2|^3}. \tag{1.16}$$

The last expression is related to the GLN, which is a link invariant that quantifies the number of times one curve intersects the surface of another in an oriented way. We will demonstrate this fact later in this work. Now, a question arises: do more complex link invariants than GLN exist that can be obtained from the on-shell action of some topological theory? The answer is affirmative, and in the next chapter we will work with the non-Abelian Chern-Simons theory coupled with Wong particles, and we will explore the link invariants associated with it.

Chapter 2

Non-Abelian Chern-Simons Theory

As we saw in the previous chapter, the only link invariant that can be obtained through the Abelian Chern-Simons theory is the GLN. Throughout this chapter, which is based on the work of L. Leal [2] and E. Fuenmayor [5], we will delve into the generalities and link invariants that can be derived from the non-Abelian Chern - Simons theory coupled with Wong particles, which carry chromo-electric charge, analogous to those appearing in QCD. As we will demonstrate later, interpreting the link invariant of the complete on-shell action is a nontrivial task. Therefore, we propose a perturbative analysis, allowing us to obtain distinct contributions of link invariants at different orders.

2.1 Generalities

The action of the Chern-Simons-Wong theory is given by:

$$S = S_{CS} + S_{int}, \tag{2.1}$$

$$S_{CS} = -\Lambda^{-1} \int d^3x \varepsilon^{\mu\nu\rho} \operatorname{Tr} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right), \tag{2.2}$$

$$S_{int} = \sum_{i=1}^{n} \int_{\gamma_i} d\tau \operatorname{Tr} \left(K_i g_i^{-1}(\tau) D_\tau g_i(\tau) \right), \tag{2.3}$$

where S_{CS} is the Chern-Simons action for $SU(N)$, S_{int} corresponds to the field-particle interaction of n Wong particles, γ_i corresponds to the worldline of the i-th particle with coordinates $z_i(\tau)$,

Λ is a constant, A_μ is a non-Abelian gauge field (analogous to the electromagnetic potential as it was shown in the previous chapter), $g_i(\tau)$ are matrices associated with the internal degrees of freedom of the particles (elements of $SU(N)$), $D_\tau g_i(\tau)$ is the covariant derivative of $g_i(\tau)$ along the worldline of the i-th particle, and K_i is a constant element in the algebra related to the initial value of the chromoelectric charge $I_i(\tau)$ defined by

$$I_i(\tau) \equiv g_i(\tau) K_i g_i^{-1}(\tau) = I_i^a T^a. \tag{2.4}$$

In the previous expression $K_i \equiv K_i^a T^a$, where T^a are the $N^2 - 1$ generators of the group algebra. Note that $I_i(\tau)$ is an element of $SU(N)$. It is well known that the Chern-Simons action is gauge invariant if the field A_μ transforms as follows

$$A_\mu \to A_\mu^\Omega = \Omega^{-1} A_\mu \Omega + \Omega^{-1} \partial_\mu \Omega. \tag{2.5}$$

The action S_{int} is gauge invariant if it satisfies

$$K_i \to K_i^\Omega = K_i, \tag{2.6}$$

$$g_i \to g_i^\Omega = \Omega^{-1} g_i, \tag{2.7}$$

$$I_i^\Omega = \Omega^{-1} I_i \Omega. \tag{2.8}$$

We define the covariant derivative of $g_i(\tau)$ as

$$D_\tau g_i(\tau) = \dot{g}_i(\tau) + A_i(\tau) g_i(\tau). \tag{2.9}$$

Note that the operator D_τ transforms as one would expect from a covariant derivative

$$(D_\tau g_i)^\Omega = \Omega^{-1} D_\tau g_i. \tag{2.10}$$

In addition, we mention that we are using the conventions and notation

$$\mathrm{Tr}\left(T^a T^b\right) = -\frac{1}{2}\delta^{ab}, \tag{2.11}$$

$$T^a T^b = f^{abc} T^c, \tag{2.12}$$

$$A_\mu = A_\mu^a T^a, \tag{2.13}$$

$$A_i = A_\mu\left(z_i(\tau)\right)\dot{z}_i^\mu(\tau), \tag{2.14}$$

where f^{abc} is the structure constant of the algebra, δ^{ab} is the Kronecker delta, A_μ^a are Abelian gauge fields, and $\dot{z}_i^\mu(\tau)$ is the velocity of the i-th Wong particle.

To take variations with respect to $A_\mu^a(x)$, it is convenient to rewrite S_{int} as

$$\begin{aligned} S_{int} =& \sum_{i=1}^{n}\int_{\gamma_i} d\tau\dot{z}_i^\mu(\tau)\left(\mathrm{tr}\left(K_i g_i^{-1}\partial_\mu g_i\right) + \mathrm{tr}\left(K_i g_i^{-1}A_\mu\left(z_i\right)g_i\right)\right) \\ =& \sum_{i=1}^{n}\int_{\gamma_i} d\tau\dot{z}_i^\mu(\tau)\,\mathrm{tr}\left(K_i g_i^{-1}\partial_\mu g_i\right) \\ &+ \int d^3x\sum_{i=1}^{n}\int_{\gamma_i} d\tau\dot{z}_i^\mu(\tau)\delta^3\left(x - z_i(\tau)\,\mathrm{tr}\left(I_i(\tau)A_\mu\left(z_i(\tau)\right)\right)\right). \end{aligned} \tag{2.15}$$

When applying the principle of least action to (2.1), we obtain the equation of motion for the field

$$\varepsilon^{\mu\nu\rho}F_{\nu\rho} = \Lambda J^\mu, \tag{2.16}$$

$$J^\mu(x) = \sum_{i=1}^{n}\int_{\gamma_i} d\tau\dot{z}_i^\mu(\tau)I_i(\tau)\delta^3\left(x - z_i(\tau)\right), \tag{2.17}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right], \tag{2.18}$$

where $F_{\mu\nu}$ is the field strength tensor, and J^μ is the current density. To vary the action (2.1) with respect to the internal variables, we must proceed with caution, as they are matrices of $SU(N)$, whose matrix elements are not independent. We use the following parameterization

$$g_i = g_i\left(\xi_i\right) = e^{\xi_i^a T^a}. \tag{2.19}$$

The variation of S_{int} with respect to the N^2-1 (multiplied by the number of Wong particles) independent parameters ξ_i^a leads to the Euler-Lagrange equations:

$$\frac{\partial L}{\partial \xi_i^a} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\xi}_i^a}\right) = 0, \tag{2.20}$$

where

$$\begin{aligned} L \equiv& \sum_{i=1}^{n}\mathrm{tr}\left(K_i g_i^{-1}(\tau)D_\tau g_i(\tau)\right) \\ =& \sum_{i=1}^{n}\mathrm{tr}\left(K_i g_i^{-1}\left(\dot{g}_i + A_i g_i\right)\right). \end{aligned} \tag{2.21}$$

Therefore, by replacing (2.21) into (2.20), the equation of motion for the chromo-electric current of the i-th particle is given by:

$$D_\tau I_i \equiv \dot{I}_i + [A_i, I_i] = 0. \tag{2.22}$$

It is possible to formally integrate this equation to obtain the general solution

$$I_i(\tau) = U_i(\tau) I_i(0) U_i^{-1}(\tau), \tag{2.23}$$

where

$$U_i(\tau) = \mathbf{T}\exp\left\{-\int_0^\tau A_i\left(\tau'\right)d\tau'\right\}. \tag{2.24}$$

Indeed, note that

$$\begin{aligned}\dot{U}_i(\tau) &= \lim_{\epsilon\to 0}\frac{\left(1 - A_i(\tau+\epsilon)\epsilon\right)U_i(\tau) - U_i(\tau)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\left(-A_i(\tau+\epsilon)U_i(\tau)\right) = -A_i(\tau)U_i(\tau) \quad\Rightarrow\quad D_\tau U_i(\tau) = 0,\end{aligned} \tag{2.25}$$

from where

$$\frac{d}{d\tau}\left(U_i^{-1}(\tau)\right) = U_i^{-1}(\tau)A_i(\tau). \tag{2.26}$$

The equations of motion for the field A_μ must satisfy the following consistency condition. Note that by taking the covariant derivative of equation (2.16)

$$\varepsilon^{\mu\nu\rho}D_\mu F_{\nu\rho} = \Lambda D_\mu J^\mu, \tag{2.27}$$

by using the Bianchi identities

$$\varepsilon^{\mu\nu\rho}D_\mu F_{\nu\rho} = 0, \tag{2.28}$$

we arrive at

$$D_\mu J^\mu = 0. \tag{2.29}$$

Now, the above condition can be written as

$$\sum_i\int d\tau\left(\frac{\partial}{\partial x^\mu}\delta^3\left(x - z_i(\tau)\right)\dot{z}_i^\mu(\tau)I_i(\tau) + \delta^3\left(x - z_i(\tau)\dot{z}_i^\mu(\tau)\left[A_\mu(x), I_i(\tau)\right]\right)\right) = 0. \tag{2.30}$$

The first term of equation (2.30) is equivalent to

$$\sum_i\left(-\int_0^T d\tau\frac{d}{d\tau}\left\{\delta^3\left(x - z_i(\tau)\right)I_i(\tau)\right\} + \int_0^T d\tau\delta^3\left(x - z_i(0)\right)\right), \tag{2.31}$$

and, in turn, the first term of equation (2.31) is equivalent to:

$$\sum_i \left(-I_i(T)\delta^3\left(x - z_i(T)\right) + I_i(0)\delta^3\left(x - z_i(0)\right)\right). \tag{2.32}$$

If we demand that $I_i(T) = I_i(0)$ (remember that $z_i(T) = z_i(0)$), then (2.32) vanishes. Substituting (2.31) into (2.30), we have that $D_\mu J^\mu$ leads to:

$$\sum_i \int_0^T d\tau \delta^3\left(x - z_i(\tau)\right) D_\tau I_i = 0. \tag{2.33}$$

Thus, the previous equation is satisfied because $D_\tau I_i = 0$ (the equation of motion for the chromo-electric current, produced by taking variations with respect to the internal variables ξ_i^a). Therefore, the Chern-Simons-Wong theory is consistent.

The action (2.1) is gauge invariant and independent of the metric, meaning it is a topological action. This aspect will serve as the foundation for the perturbative scheme in order to obtain link invariants, which will be discussed in the next section.

2.2 Perturbative approach

Now we will focus our attention on the perturbative development of the model. The equations of motion (2.16) and (2.22) constitute a highly nonlinear system of equations for A_μ^a and I_i^a, for which we do not intend to obtain exact solutions. Nevertheless, let us assume that such solutions exist and that under certain boundary conditions, we can derive the potential A_μ^a as a functional of the curves γ_i (already given, which are living on space-time) that constitute the current J^μ in equation (2.16), namely

$$A_\mu^a = A_\mu^a\left[\gamma_i\right]. \tag{2.34}$$

The equation to solve would then be

$$\varepsilon^{\mu\nu\rho} F_{\nu\rho} = \Lambda \sum_{i=1}^{n} \int_{\gamma_i} d\tau \dot{z}_i^\mu(\tau) I_i[A] \delta^3\left(x - z_i(\tau)\right), \tag{2.35}$$

and its solutions are expected to be functionals of the curves that underlie the currents, as implied by equation (2.34). By substituting $A_\mu^a\left[\gamma_i\right]$ into equation (2.1), we can then express the action S as a functional of the curves γ_i, denoted as

$$S = S[\gamma_i], \tag{2.36}$$

effectively eliminating all dependence on A_μ^a. Given that the action is independent of the metric, the action on the equations of motion $S[\gamma_i]$ stands as a topological invariant. However, it is a topological invariant that hinges on a collection of curves, precisely characterizing it as a knot invariant (or more accurately, a link invariant: a link being a collection of knots).

Hence, we have the opportunity to generate analytical expressions for link invariants through the solutions of the classical Chern-Simons-Wong equations and the computation of the action on the equations of motion (on-shell action). However, given the intricate nature of the equations at hand, the aforementioned seems to be nothing more than a mere possibility, unless an approximate solution scheme is embraced, such that we preserve the invariance under diffeomorphisms of the problem. This scheme can be outlined as follows: equation (2.35) can be solved perturbatively, expanding in powers of the parameter Λ. Substituting such a perturbative solution into the action yields the on-shell action expressed as a series of powers in Λ:

$$S_{on-\text{ shell}}\left([\gamma];\Lambda\right)=\sum_{p=0}^{\infty}\Lambda^{p}S^{(p)}[\gamma], \tag{2.37}$$

where $S^{(p)}[\gamma]$ is the p-th coefficient of the expansion. Now, if $S_{on-shell}$ is a link invariant, then its derivatives with respect to Λ must also be link invariants. Thus, it follows that the coefficients $S^{(p)}[\gamma]$ are link invariants. A practical implication of this straightforward argument is that obtaining the complete series (2.37) is not necessary to derive knot invariants.

Now, in order to calculate $S_{on-shell}$, note that (here we omit the index i for convenience) if we combine equations (2.4) and (2.23), we have

$$\begin{aligned} I(\tau) &= U(\tau)g(0)Kg^{-1}(0)U^{-1}(\tau) \\ &= g(\tau)Kg^{-1}(\tau). \end{aligned} \tag{2.38}$$

Recall that $g(\tau)=U(\tau)g(0)$ and using the equation (2.25):

$$D_{\tau}g(\tau)=0 \;\;\Rightarrow\;\; D_{\tau}I_i(\tau)=0, \tag{2.39}$$

then

$$S_{\text{int}}\left.\right|_{\text{on-shell}}=0. \tag{2.40}$$

Therefore, it only remains to consider $\left.S_{CS}\right|_{\text{on-shell}}$. Note that

$$\dot{I}_i+[A_i,I_i]=0 \quad\Rightarrow\quad \frac{dI_i^a(\tau)}{d\tau}+\Lambda R_i^{ac}(\tau)I_i^c(\tau)=0, \tag{2.41}$$

where

$$R_i^{ac}\equiv f^{abc}\dot{z}_i^{\mu}a_{\mu}^{b}\left(z_i\right), \tag{2.42}$$

$$a_{\mu}\equiv\Lambda^{-1}A_{\mu}, \tag{2.43}$$

and the solution to equation (2.41) is

$$\overrightarrow{I_i}(\tau) = \mathbf{T}\exp\left[-\Lambda\int_0^\tau d\tau' R_i(\tau')\right]\overrightarrow{I_i}(0), \tag{2.44}$$

where $\vec{I_i}$ is the vector of N^2-1 components I_i^a, and R_i is the matrix of $(N^2-1)^2$ elements R_i^{ab} given in (2.42). By performing a Taylor series expansion, the equations of motion (2.16) for the field A_μ can be written as

$$\begin{aligned} 2\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho^a(x) = & -\Lambda\varepsilon^{\mu\nu\rho}f^{abc}a_\nu^b(x)a_\rho^c(x) + \sum_{i=1}^n \oint_{\gamma_i} dz^\mu \delta^3(x-z)I_i^a(0) \\ & -\Lambda\sum_{i=1}^n \oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1} R_{\mu_1}^{aa_1}(z_1)\,\delta^3(x-z)I_i^{a_1}(0) \\ & +\Lambda^2\sum_{i=1}^n \oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1}\int_0^{z_1} dz_2^{\mu_2} R_{\mu_1}^{aa_1}(z_1)\,R_{\mu_2}^{a_1a_2}(z_2) \\ & \delta^3(x-z)I_i^{a_2}(0) \\ & \vdots \\ & +(-\Lambda)^p\sum_{i=1}^n \oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1}\dots\int_0^{z_{p-1}} dz_p^{\mu_p} R_{\mu_1}^{aa_1}(z_1) \\ & \dots R_{\mu_p}^{a_{p-1}a_p}(z_p)\,\delta^3(x-z)I_i^{a_p}(0) \\ & \vdots\ . \end{aligned} \tag{2.45}$$

If we introduce the power series

$$a_\mu^a(x) = \sum_{p=0}^\infty \Lambda^p a_\mu^{(p)a}(x), \tag{2.46}$$

from where the p-th order term reads

$$\begin{aligned} 2\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho^{(p)a}(x) = & -\varepsilon^{\mu\nu\rho}f^{abc}\sum_{r,s=0}^{r+s=p-1} a_\nu^{(r)b}a_\rho^{(s)c} + \\ & +\sum_{r=1}^p(-1)^r\sum_{i=1}^n\oint_{\gamma_i} dz^\mu\int_0^z dz_1^{\mu_1}\dots\int_0^{z_{r-1}} dz_r^{\mu_r}\sum_{s_1,\dots,s_r=o}^{s_1+\dots+s_r=p-r} R_{\mu_1}^{(s_1)aa_1}(z_1)\times \\ & \times R_{\mu_2}^{(s_2)a_1a_2}(z_2)\quad\dots\quad R_{\mu_r}^{(s_r)a_{r-1}a_r}(z_r)\,I_i^{a_r}(0)\delta^3(x-z), \end{aligned} \tag{2.47}$$

for $p \geq 1$. If $p=0$, the corresponding equation is

$$2\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho^{(0)a} = \sum_{i=1}^n\oint_{\gamma_i} dz_i^\mu \delta^3(x-z_i)\,I_i^a(0). \tag{2.48}$$

Note that these equations have the form of Ampere's Law

$$\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho^{(p)a}(x) = J^{(p)\mu a}(x), \tag{2.49}$$

so their solution is expressed as the Biot-Savart Law

$$a^{(p)a}(x) = -\frac{1}{4\pi}\int d^3x'\varepsilon_{\alpha\beta\gamma}J^{(p)\beta a}(x')\frac{(x-x')^\gamma}{|x-x'|^3} + \partial_\alpha f^a(x), \tag{2.50}$$

where $f^a(x)$ is arbitrary. Thus, the on-shell action will be written as

$$\begin{aligned} S_{\text{on-shell}} &= S_{\text{on-shell}}^{CS} \\ &= \frac{\Lambda}{2}\int d^3x\varepsilon^{\mu\nu\rho}\left(a_\mu^a\partial_\nu a_\rho^a + \frac{\Lambda}{3}f^{abc}a_\mu^a a_\nu^b a_\rho^c\right)\Big|_{\text{on-shell}} \\ &= \frac{\Lambda}{2}\sum_{p=0}^{\infty}S^{(p)}\Lambda^p, \end{aligned} \tag{2.51}$$

where

$$S^{(p)} = \int d^3x\varepsilon^{\mu\nu\rho}\left(\sum_{r,s}^{r+s=p}\left(a_\mu^{(r)a}\partial_\nu a_\mu^{(s)a}\right) + \frac{1}{3}f^{abc}\sum_{r,s,q}^{r+s+q=p-1}\left(a_\mu^{(r)a}a_\nu^{(s)b}a_\rho^{(q)c}\right)\right). \tag{2.52}$$

Now, we are in a position to determine, in principle, the link invariants up to order p arising from each term in the perturbative expansion of the on-shell action. In this study, we will compute contributions up to order 0, 1, and 2, which correspond to analytical expressions that characterize the GLN, the TMC (related to the Borromean rings), and the four-components link. We will demonstrate these results in the chapter 4.

2.3 Zeroth-order contribution

At zeroth-order we have

$$S^{(0)} = \int d^3x\varepsilon^{\mu\nu\rho}a_\mu^{(0)a}\partial_\nu a_\rho^{(0)a}, \tag{2.53}$$

$$a_\mu^{(0)a}(x) = \frac{1}{2}\sum_{i=1}^{n}D_{i\mu}(x)I_i^a(0), \tag{2.54}$$

$$D_{i\mu}(x) \equiv \frac{1}{4\pi}\oint_{\gamma_i}dz^\rho\frac{(x-z)^\nu}{|x-z|^3}\varepsilon_{\mu\nu\rho}. \tag{2.55}$$

Note that the gradient that appears in equation (2.50) does not contribute to the on-shell action, so we can discard it. By replacing (2.54) and (2.55) in (2.53), we have

$$S^{(0)} = \frac{1}{4}\sum_{i,j} I_i^a(0) I_j^a(0) L(i,j), \tag{2.56}$$

where

$$L(i,j) \equiv \frac{1}{4\pi}\oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\rho \frac{(z-y)^\beta}{|z-y|^3}\varepsilon_{\mu\nu\rho} \tag{2.57}$$

is the GLN, which also appears in (1.16). As was mentioned in the previous chapter, this link invariant measures the number of times that a curve intersects the surface of another. In Figure 2.1 can be seen a pair of curves with a GLN equal to one, because the blue curve intersects the surface of the red one once. Also, in Figure 2.2 we have a pair of curves with a GLN equal to zero, because the blue curve intersects in one way and then in an opposite way the surface of the red one.

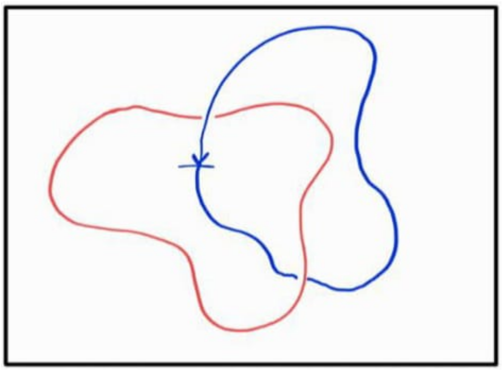

Figure 2.1: Curves with GLN equal one.

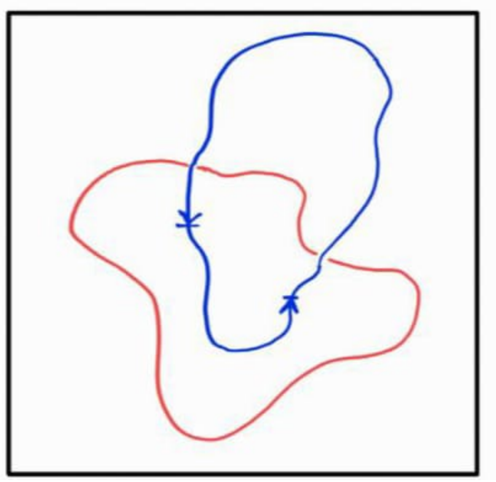

Figure 2.2: Trivial link (Not linked).

2.4 First-order contribution

The first-order on-shell action is given by:

$$S^{(1)} = \int d^3x \varepsilon^{\mu\nu\rho}\left(2a_\mu^{(0)a}\partial_\nu a_\rho^{(1)a} + \frac{1}{3}f^{abc}\left(a_\mu^{(0)a}a_\nu^{(0)b}a_\rho^{(0)c}\right)\right). \tag{2.58}$$

Note that to calculate this expression, it is not necessary to know the specific value of $a_\rho^{(1)a}$.

It suffices to know its curl, which is given by

$$\begin{aligned}\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho^{(1)}(x) = &-\frac{1}{2}\varepsilon^{\mu\nu\rho} f^{abc} a_\nu^{(0)}(x) a_\rho^{(0)}(x)\\ &-\frac{1}{2}\sum_{i=1}^{n}\oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1} R_{\mu_1}^{(0)aa_1}(z_1)\,\delta^3(x-z) I_i^{a_1}(0).\end{aligned} \tag{2.59}$$

By replacing (2.54) and (2.59) into (2.58), we have

$$\begin{aligned}S^{(1)} = -\frac{1}{4}\sum_{i,j,k} f^{abc} I_i^a(0) I_j^b(0) I_k^c(0) \Bigg\{ &\frac{1}{3}\int d^3x \varepsilon^{\mu\nu\rho} D_{i\mu}(x) D_{j\nu}(x) D_{k\rho}(x) + \\ &+ \oint_{\gamma_i} dz^\mu \int_0^z dy^\nu D_{j\mu}(z) D_{k\nu}(y) \Bigg\}.\end{aligned} \tag{2.60}$$

The factor $f^{abc} I_i^a(0) I_j^b(0) I_k^c(0)$ in the above expression vanishes when the currents $I_i^a(0)$, $I_j^b(0)$, and $I_k^c(0)$ are linearly dependent. As a consequence of this, $S^{(1)}$ is zero when the current consists of only one or two Wong particles. To interpret the expression for $S^{(1)}$, let us consider the (nontrivial) simplest case: gauge group $SU(2)$ and three particles with orthonormal isovectors, $I_i^a(0) = \delta_i^a$. In this case

$$\begin{aligned}S^{(1)}(1,2,3) = &-\frac{1}{2}\int d^3x \varepsilon^{\mu\nu\rho} D_{1\mu}(x) D_{2\nu}(x) D_{3\rho}(x)\\ &-\frac{1}{2}\int d^3x \int d^3y \Big\{ T_1^{[\mu x,\nu y]} D_{2\mu}(x) D_{3\nu}(y) + \\ &+ T_2^{[\mu x,\nu y]} D_{3\mu}(x) D_{1\nu}(y) + \\ &+ T_3^{[\mu x,\nu y]} D_{1\mu}(x) D_{2\nu}(y) \Big\},\end{aligned} \tag{2.61}$$

where we have introduced the bilocal object $T_{\gamma_i}^{\mu x,\nu y}$ associated with the curve γ_i

$$T_{\gamma_i}^{\mu x,\nu y} \equiv \oint_{\gamma_i} dz^\mu \int_0^z dz'^\nu \delta^3(x-z)\delta^3(y-z'), \tag{2.62}$$

and the definition

$$T_{\gamma_i}^{[\mu x,\nu y]} \equiv \frac{1}{2}\left(T_{\gamma_i}^{\mu x,\nu y} - T_{\gamma_i}^{\nu x,\mu y}\right). \tag{2.63}$$

The expression (2.61) turns out to be, except for a factor, the TMC $\bar{\mu}(1,2,3)$, which is related to three linked curves in space-time, such that the GLN between them is always zero (this condition will be necessary for the consistency of our theory). For example, this expression detects the linking of the Borromean rings, which are shown in Figure 2.3. Note that the curve γ_i (red) intersects the surface enclosed by the curves γ_j (blue) and γ_k (green), then intersects in the opposite direction to the blue and green surfaces (ensuring that the GLN between (γ_i,γ_j) and (γ_i,γ_k) is zero) to return to the initial point. A similar description of this process can be given for each curve with respect to the other two. Therefore, it is evident that the GLN between any pair of curves of the Borromean rings is always zero .

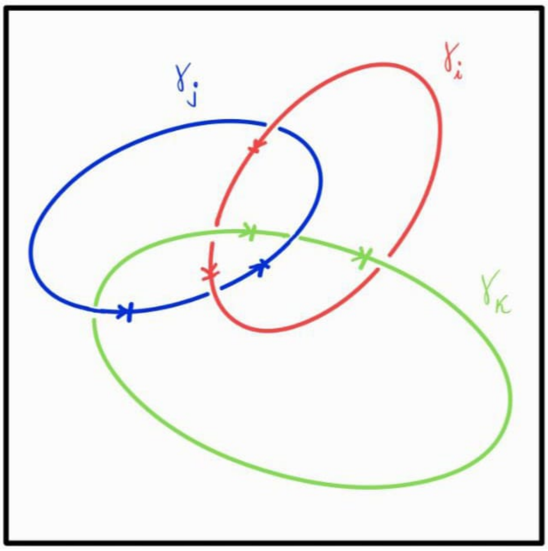

Figure 2.3: Borromean Rings (Note that the GLN between any pair of curves is zero.)

Indeed, the TMC follows the GLN in an infinite sequence of link invariants discovered by Milnor: the so-called "Higher-Order Linking Coefficients". The n-th coefficient is defined if all previous ones are zero. This result naturally emerges in our theory (as the consistency conditions for this perturbative analysis) and will be demonstrated at the end of this chapter.

In order to continue our perturbative analysis and obtain the link invariant associated with the on-shell action to second order, it is necessary to introduce certain conventions that allow us to perform calculations more efficiently. In the next section, we will introduce the cycle space, Loop coordinates, and a generalization of the Einstein summation convention for continuous variables, which will be useful for simplifying calculations and providing an appropriate geometric interpretation of the link invariants we will derive.

2.5 Loop coordinates and cycle space

We will denote the dependence of a tensorial function on a continuous variable by placing an index indicating that variable as follows

$$A_{\mu\nu\ldots\rho}(x, y, \ldots, z) \equiv A_{\mu x\ \nu y\ \ldots\ \rho z}. \tag{2.64}$$

The aforementioned relation will allow us to establish a form of a generalized Einstein convention applicable to discrete-continuous variables. In this context, tensorial functions are integrated over the repeated continuous variable, while also, as usually, summing over the repeated discrete variables; for example,

$$A_{\mu x}B^{\mu x\ \nu y\cdots} \equiv \sum_{\mu}\int A_{\mu x}B^{\mu x\ \nu y\cdots}d^3x = \sum_{\mu}\int A_{\mu}(x)B^{\mu\nu\cdots}(x,y\ldots)d^3x. \tag{2.65}$$

When repeated indices that are not integrated appear, we will place a "bar" above the letter indicating that variable in the following way:

$$A_{\mu x\ \nu\bar{y}}B^{\mu x\ \nu\bar{y}} \equiv \sum_{\mu}\int A_{\mu x\ \nu\bar{y}}B^{\mu x\ \nu\bar{y}}d^3x. \tag{2.66}$$

Many times it will be convenient, for an even greater simplification, to indicate the set of "discrete-continuous" indices by a single index, which we will denote by a lowercase Latin letter, as follows:

$$A^{\mu x\ \nu y\ \ldots\ \rho z} \equiv A^{ab\ldots c}. \tag{2.67}$$

At this point, it should be emphasized that the T-objects that appear in the equation (2.62) are a particular case of an infinite family of objects given by

$$\begin{aligned} T_i^{\mu_1 x_1 \mu_2 x_2 \ldots \mu_n x_n} \equiv \oint_{\gamma_i} dz^{\mu_1}\int_0^{z} dz_1^{\mu_2}\int_0^{z_1} dz_2^{\mu_3}\ldots\int_0^{z_{n-2}} dz_{n-1}^{\mu_n}\delta^{(3)}\left(x_1 - z\right)\times \\ \delta^{(3)}\left(x_2 - z_1\right)\times\delta^{(3)}\left(x_3 - z_2\right)\ldots\delta^{(3)}\left(x_n - z_{n-1}\right), \end{aligned} \tag{2.68}$$

such objects are called loop coordinates, and where introduced in [7]. The above relation defines the distributional T-objects, of rank n, also known as multi-tangents of cycles or multi-tangents of paths. It is directly observed, from the definition (2.68), that the T with one-index is nothing more than the form factor or tangent distribution of the closed curve γ_i,

$$T_i^{\mu x} = \oint_{\gamma_i} dy^{\mu}\delta^{(3)}(x-y), \tag{2.69}$$

and the one with two-indices is the same bi-local object defined in (2.62). The multi-tangents contain all the relevant information needed to determine any element in the extended cycle space. Moreover, they possess the property of determining the Wilson line element,

$$W_A(\gamma) = \mathrm{Tr}\left[e^{\oint_{\gamma} A_a dy^a}\right],$$

for any connection. For these reasons, they can be considered as promising candidates for defining geometric coordinates. However, they do not constitute independent variables; in fact, they obey a set of constraints:

$$\frac{\partial}{\partial x_i^{\mu_i}}T_j^{\mu_1 x_1\cdots\mu_i x_i\cdots\mu_n x_n} = \left(-\delta\left(x_i - x_{i-1}\right) + \delta\left(x_i - x_{i+1}\right)\right)T_j^{\mu_1 x_1\cdots\mu_{i-1}x_{i-1}\mu_{i+1}x_{i+1}\cdots\mu_n x_n}, \tag{2.70}$$

which is called the differential constraint [7]. The points x_0 and x_{n+1} correspond to the starting values of the cycle or closed path, and the Dirac deltas are defined in the same dimension as the manifold where the paths "live",

$$T_i^{\{\mu_1\cdots\mu_k\}\mu_{k+1}\cdots\mu_n} = \sum_{P_k} T_i^{P_k(\mu_1\cdots\mu_n)} = T_i^{\mu_1\cdots\mu_k} T_i^{\mu_{k+1}\cdots\mu_n}, \tag{2.71}$$

which is called the algebraic constraint [7]. The sum is taken over all permutations of the variables μ that preserve the ordering of $\mu_1, \cdots, \mu_k$ and $\mu_{k+1}, \cdots, \mu_n$ among themselves.

Now, we will introduce the object $g_{\mu x\ \nu y}$ (symmetric in its pairs of indices), and it is defined as follows:

$$g_{\mu x\ \nu y} \equiv -\frac{1}{4\pi}\varepsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}. \tag{2.72}$$

This, together with $g^{\mu x\ \nu y}$, defined by,

$$g^{\mu x\ \nu y} \equiv \varepsilon^{\mu\nu\rho}\partial_\rho\delta(x-y), \tag{2.73}$$

naturally appears in the solution of the differential constraint (2.70) obeyed by the T-objects and constitute a metric in the space of transverse rank-one vector densities. Note that these conventions are useful and allow us to write known quantities in a compact way. For example, the D-objects (2.55) that appear when solving the equations of motion of the perturbative analysis for the non-Abelian theory

$$D_\mu(x,\gamma) \equiv D_{i\ \mu x} = -g_{\mu x\ \nu y}T_i^{\nu y}. \tag{2.74}$$

Now, the GLN reads

$$L(i,j) = \frac{1}{4\pi}\oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\rho \frac{(z-y)^\nu}{|z-y|^3}\varepsilon_{\mu\nu\rho} = T^{\mu x} g_{\mu x\ \nu y} T^{\nu y}. \tag{2.75}$$

Similarly, the Biot-Savart law of the non-Abelian theory

$$a_\mu^{(p)a}(x) = \frac{1}{4\pi}\int d^3x' \varepsilon_{\mu\nu\rho} J^{(p)a\ \nu}(x')\frac{(x-x')^\rho}{|x-x'|^3} = -J^{(p)a\ \nu y} g_{\mu x\ \nu y}. \tag{2.76}$$

Finally, the equations of motion of the non-Abelian theory (formally the same as Ampere's Law) can be written as

$$g^{\mu x\ \nu y} a_{\nu y}^{(p)a} = -\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho^{(p)a} = -J^{(p)a\ \mu x}. \tag{2.77}$$

Note that the g-objects raise and lower indices, as one would expect from a metric tensor.

In order to further simplify our problem and interpret all our results later, we will consider the simplest nontrivial case, which consists of restricting the gauge group to $SU(2)$. Our general structure constants (which previously belonged to any arbitrary group) are, in this case, the totally antisymmetric Levi-Civita symbols: $f^{abc} \leftrightarrow \varepsilon^{abc}$. We will denote with an "arrow" (indicating that they are vectors) any quantity that has components within the internal space; that is, iso-vectors will be indicated by "arrows". Thus, for example, the field $a_\mu^{(p)a}(x)$ will be written as $\vec{a}_\mu^{(p)}(x)$, the currents $I_i^a(0)$ will be written as $\vec{I}_i$, etc. Now, thanks to the conventions and notation developed throughout this section, we are in a position to "straightforwardly" calculate the second order on-shell action, which we will do in the following section.

2.6 Second-order contribution

Using the conventions introduced in the preceding section, the second-order on-shell action is given by

$$S^{(2)} = \varepsilon^{\mu\nu\rho}\left[2\partial_\nu \vec{a}^{(2)}_{\rho x}\cdot\vec{a}^{(0)}_{\mu x} + \partial_\nu \vec{a}^{(1)}_{\rho x}\cdot\vec{a}^{(1)}_{\mu x} + \vec{a}^{(1)}_{\mu x}\cdot\left(\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x}\right)\right], \tag{2.78}$$

where

$$\begin{aligned} 2\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}^{(2)}_{\rho x} = -2g^{\mu x\ \nu y}\vec{a}^{(2)}_{\nu y} =& -2\varepsilon^{\mu\nu\rho}\vec{a}^{(1)}_{\nu\bar{x}}\times\vec{a}^{(0)}_{\rho\bar{x}} - \sum_i T_i^{\mu x\ \mu_1 x_1}\vec{a}^{(1)}_{\mu_1 x_1}\times\vec{I}_i \\ &+ \sum_i T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\vec{a}^{(0)}_{\mu_1 x_1}\times\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_i\right), \end{aligned} \tag{2.79}$$

$$\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}^{(1)}_{\rho x} = \vec{J}^{(1)\mu x} = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x} - \frac{1}{2}\sum_i T_i^{\mu x\ \nu y}\vec{a}^{(0)}_{\nu y}\times\vec{I}_i, \tag{2.80}$$

$$\vec{a}^{(1)}_{\mu x} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma}\left(\vec{a}^{(0)}_{\beta y}\times\vec{a}^{(0)}_{\gamma y}\right)g_{\mu x\ \alpha y} + \frac{1}{2}\sum_i g_{\mu x\ \alpha y}T_i^{\alpha y\ \mu_1 x_1}\left(\vec{a}^{(0)}_{\mu_1 x_1}\times\vec{I}_i\right). \tag{2.81}$$

Now, our interest lies in expressing the equation (2.78) in terms of the fields $\vec{a}^{(0)}_{\mu x}$. Note that the first term in the equation (2.78) is given by

$$\begin{aligned} \varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}^{(2)}_{\rho x}\cdot\vec{a}^{(0)}_{\mu x} =& -\varepsilon^{\mu\nu\rho}\left(\vec{a}^{(0)}_{\nu\bar{x}}\times\vec{a}^{(1)}_{\rho\bar{x}}\right)\cdot\vec{a}^{(0)}_{\mu x} - \frac{1}{2}\sum_i T_i^{\mu x\ \mu_1 x_1}\left(\vec{a}^{(1)}_{\mu_1 x_1}\times\vec{I}_i\right)\cdot\vec{a}^{(0)}_{\mu x} \\ &+ \frac{1}{2}\sum_i T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{a}^{(0)}_{\mu_1 x_1}\times\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\vec{a}^{(0)}_{\mu x}. \end{aligned} \tag{2.82}$$

If we replace the equation (2.81) for the field $\vec{a}^{(1)}_{\mu x}$ into (2.82), we have

$$
\begin{aligned}
2\varepsilon^{\mu\nu\rho}\partial_\nu \vec{a}^{(2)}_{\rho x}\cdot\vec{a}^{(0)}_{\mu x} &= -\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\rho x\ \alpha y}\left[\vec{a}^{(0)}_{\nu x}\times\left(\vec{a}^{(0)}_{\beta y}\times\vec{a}^{(0)}_{\gamma y}\right)\right]\cdot\vec{a}^{(0)}_{\mu x} \\
&\quad - \sum_i \varepsilon^{\mu\nu\rho} g_{\rho x\ \alpha y} T_i^{\alpha y\ \mu_1 x_1}\left[\vec{a}^{(0)}_{\nu x}\times\left(\vec{a}^{(0)}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\vec{a}^{(0)}_{\mu x} \\
&\quad - \frac{1}{2}\sum_i \varepsilon^{\alpha\beta\gamma} g_{\mu_1 x_1\ \alpha y} T_i^{\mu x\ \mu_1 x_1}\left[\left(\vec{a}^{(0)}_{\beta y}\times\vec{a}^{(0)}_{\gamma y}\right)\times\vec{I}_i\right]\cdot\vec{a}^{(0)}_{\mu x} \\
&\quad - \frac{1}{2}\sum_{i,j} g_{\mu_1 x_1\ \alpha y} T_i^{\mu x\ \mu_1 x_1} T_j^{\alpha y\ \mu_2 x_2}\left[\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_j\right)\times\vec{I}_i\right]\cdot\vec{a}^{(0)}_{\mu x} \\
&\quad + \sum_i T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{a}^{(0)}_{\mu_1 x_1}\times\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\vec{a}^{(0)}_{\mu x}.
\end{aligned} \tag{2.83}
$$

The second and third term in (2.78) can be written as

$$
\begin{aligned}
\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}^{(1)}_{\rho x}\cdot\vec{a}^{(1)}_{\mu x} &= -\frac{1}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\mu x\ \alpha y}\left(\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x}\right)\cdot\left(\vec{a}^{(0)}_{\beta y}\times\vec{a}^{(0)}_{\gamma y}\right) \\
&\quad - \frac{1}{2}\sum_i \varepsilon^{\mu\nu\rho} g_{\mu x\ \alpha y} T^{\alpha y\ \mu_1 x_1}\left(\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x}\right)\cdot\left(\vec{a}^{(0)}_{\mu_1 x_1}\times\vec{I}_i\right) \\
&\quad - \frac{1}{4}\sum_{i,j} g_{\mu x\ \alpha y} T_i^{\mu x\ \mu_1 x_1} T_j^{\alpha y\ \mu_2 x_2}\left(\vec{a}^{(0)}_{\mu_1 x_1}\times\vec{I}_i\right)\cdot\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_j\right),
\end{aligned} \tag{2.84}
$$

$$
\begin{aligned}
\varepsilon^{\mu\nu\rho}\vec{a}^{(1)}_{\mu x}\cdot\left(\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x}\right) &= \frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\mu x\ \alpha y}\left(\vec{a}^{(0)}_{\beta y}\times\vec{a}^{(0)}_{\gamma y}\right)\cdot\left(\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x}\right) \\
&\quad + \frac{1}{2}\sum_i \varepsilon^{\mu\nu\rho} g_{\mu x\ \alpha y} T_i^{\alpha y\ \mu_1 x_1}\left(\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x}\right)\cdot\left(\vec{a}^{(0)}_{\mu_1 x_1}\times\vec{I}_i\right),
\end{aligned} \tag{2.85}
$$

If we use the following vector identities:

$$
\begin{aligned}
\left(\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x}\right)\cdot\left(\vec{a}^{(0)}_{\beta y}\times\vec{a}^{(0)}_{\gamma y}\right) &= -\left[\vec{a}^{(0)}_{\nu x}\times\left(\vec{a}^{(0)}_{\beta y}\times\vec{a}^{(0)}_{\gamma y}\right)\right]\cdot\vec{a}^{(0)}_{\rho x}, \\
\left(\vec{a}^{(0)}_{\nu x}\times\vec{a}^{(0)}_{\rho x}\right)\cdot\left(\vec{a}^{(0)}_{\mu_1 x_1}\times\vec{I}_i\right) &= -\left[\vec{a}^{(0)}_{\nu x}\times\left(\vec{a}^{(0)}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\vec{a}^{(0)}_{\rho x}, \\
\left(\vec{a}^{(0)}_{\mu_1 x_1}\times\vec{I}_i\right)\cdot\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_j\right) &= -\left[\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_j\right)\times\vec{I}_i\right]\cdot\vec{a}^{(0)}_{\mu_1 x_1},
\end{aligned} \tag{2.86}
$$

and replace (2.83), (2.84), (2.85) into (2.78), the second-order on-shell action is given by

$$
\begin{aligned}
S^{(2)} = & \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\mu x\ \alpha y}\left[\vec{a}^{(0)}_{\nu x} \times \left(\vec{a}^{(0)}_{\beta y} \times \vec{a}^{(0)}_{\gamma y}\right)\right] \cdot \vec{a}^{(0)}_{\rho x} \\
& + \sum_i \varepsilon^{\mu\nu\rho} g_{\mu x\ \alpha y}\left[\vec{a}^{(0)}_{\nu x} \times \left(\vec{a}^{(0)}_{\mu_1 x_1} \times \vec{I}_i\right)\right] \cdot \vec{a}^{(0)}_{\rho x}\left(T_i^{\alpha y\ \mu_1 x_1} - \frac{1}{2}T_i^{\mu_1 x_1\ \alpha y}\right) \\
& + \frac{1}{2}\sum_{i,j} g_{\mu x\ \alpha y} T_j^{\alpha y\ \mu_2 x_2}\left[\vec{I}_i \times \left(\vec{a}^{(0)}_{\mu_2 x_2} \times \vec{I}_j\right)\right] \cdot \vec{a}^{(0)}_{\mu_1 x_1}\left(T_i^{\mu_1 x_1\ \mu x} - \frac{1}{2}T_i^{\mu x\ \mu_1 x_1}\right) \\
& + \sum_i T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{a}^{(0)}_{\mu_1 x_1} \times \left(\vec{a}^{(0)}_{\mu_2 x_2} \times \vec{I}_i\right)\right] \cdot \vec{a}^{(0)}_{\mu x}.
\end{aligned}
\tag{2.87}
$$

Now, using our new notation, we proceed to introduce the zeroth-order field $\vec{a}^{(0)}_{\mu x}$ into the previous equation (2.87) expressed in terms of the one-index T-object,

$$
\varepsilon^{\mu\nu\rho}\partial_\nu \vec{a}^{(0)}_{\rho x} = \frac{1}{2}\sum_{i=1}^{n} T_i^{\mu x}\vec{I}_i, \tag{2.88}
$$

so we will have $S^{(2)}$ solely in terms of path coordinates and the metric. Thus, we obtain:

$$
\begin{aligned}
S^{(2)} = & \frac{1}{8}\sum_{i,j,k,l}\left(\left[\vec{I}_i \times \left(\vec{I}_j \times \vec{I}_k\right)\right] \cdot \vec{I}_l\right) g_{\mu x\ \alpha y} \\
& \Bigg\{ \frac{3}{8}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\nu x\ \mu_1 x_1} g_{\rho x\ \mu_4 x_4} g_{\beta y\ \mu_2 x_2} g_{\gamma y\ \mu_3 x_3} T_i^{\mu_1 x_1} T_j^{\mu_2 x_2} T_k^{\mu_3 x_3} T_l^{\mu_4 x_4} \\
& + \varepsilon^{\mu\nu\rho}\left(T_j^{\alpha y\ \mu_1 x_1} - \frac{1}{2}T_j^{\mu_1 x_1\ \alpha y}\right) g_{\nu x\ \mu_2 x_2} g_{\rho x\ \mu_4 x_4} g_{\mu_1 x_1\ \mu_3 x_3} T_i^{\mu_2 x_2} T_k^{\mu_3 x_3} T_l^{\mu_4 x_4} \\
& - T_j^{\alpha y\ \mu_2 x_2}\left(T_i^{\mu_1 x_1\ \mu x} - \frac{1}{2}T_i^{\mu x\ \mu_1 x_1}\right) g_{\mu_2 x_2\ \mu_3 x_3} g_{\mu_1 x_1\ \mu_4 x_4} T_k^{\mu_3 x_3} T_l^{\mu_4 x_4} \\
& + T_j^{\mu x\ \mu_1 x_1\ \mu_2 x_2} g_{\mu_1 x_1\ \mu_3 x_3} g_{\mu_2 x_2\ \mu_4 x_4} T_i^{\mu_3 x_3} T_k^{\mu_4 x_4} T_l^{\alpha y} \Bigg\}.
\end{aligned}
\tag{2.89}
$$

Finally, to obtain an expression that is as simplified as possible, we proceed to decompose the two-index T-objects into their symmetric and antisymmetric parts $\left(T_i^{ab} = T_i^{(ab)} + T_i^{[ab]}\right)$, and use the fact that the symmetric part factorizes into two one-index T-objects $\left(T_i^{(ab)} = T_i^a T_i^b\right)$ according to the algebraic constraint (2.71). When these form factors (T_i^a, T_i^b) are appropriately combined with the present metrics (g_{ab}), GLNs are formed as in equation (2.75), which we will discard as they must vanish for the physical theory to be consistent. Therefore, we obtain the following more compact form of the second order on-shell action:

$$
\begin{aligned}
S^{(2)} =& \frac{1}{8}\sum_{i,j,k,l}\left(\left[\vec{I}_i\times\left(\vec{I}_j\times\vec{I}_k\right)\right]\cdot\vec{I}_l\right)\left\{\frac{3}{8}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\nu x\ a}g_{\rho x\ d}g_{\beta y\ b}g_{\gamma y\ c}T_i^aT_j^bT_k^cT_l^d\right.\\
&+\frac{3}{2}\varepsilon^{\mu\nu\rho}T_j^{[\alpha y\ a]}g_{\nu x\ b}g_{\rho x\ d}g_{ac}T_i^bT_k^cT_l^d+\frac{3}{2}T_j^{[b\ \alpha y]}T_i^{[a\ \mu x]}g_{bc}g_{ad}T_k^cT_l^d\\
&\left.+T_j^{\mu x\ a\ b}g_{ac}g_{bd}T_i^cT_k^dT_l^{\alpha y}\right\}g_{\mu x\ \alpha y}
\end{aligned}
\tag{2.90}
$$

It can be proven that the above equation detects the linking of four closed curves in space-time, such that the GLN and the TMC are always zero among them, as shown in Figure 2.4. This proof will be carried out in the final chapter of this work, corresponding to the geometric interpretation of the link invariants obtained so far.

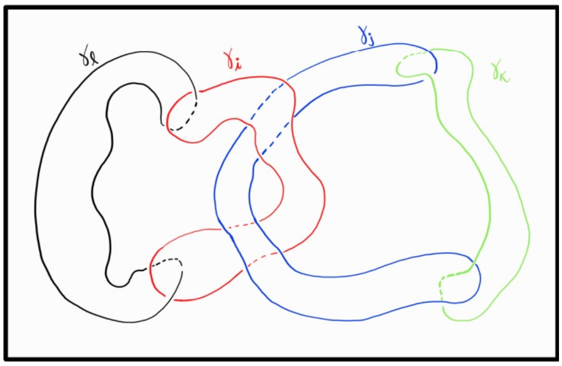

Figure 2.4: Four-components link (note that the GLN and TMC between all the curves is 0.)

Finally, in the last section of this chapter, we will demonstrate that the consistency conditions of the perturbative analysis lead to the GLN being zero for the action $S^{(1)}$, and that the GLN along with the TMC are zero for the action $S^{(2)}$. As mentioned earlier, the advantage of the introduced perturbative analysis is that, in general, we can stop at any order, and the remaining contributions necessarily vanish due to the consistency conditions.

2.7 Consistency of the theory

2.7.1 $S^{(0)}$: Gauss linking number

Now, let us check the consistency condition for the zeroth-order in our perturbative analysis. Note that, if we take the divergence of the equation of motion (2.88) for the field $\vec{a}^{(0)}_{\mu x}$, it follows that

$$
\partial_\mu T_i^{\mu x}=0,
\tag{2.91}
$$

since $\varepsilon^{\mu\nu\rho}\partial_\mu\partial_\nu\vec{a}^{(0)}_{\rho x} = 0$, so equation (2.91) holds because of the differential constraint (2.70). Thus, the theory at zeroth-order is consistent.

2.7.2 $S^{(1)}$: Milnor's third linking coefficient

If we use the equation (2.47) we can write the first-order equation of motion as follows:

$$\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}^{(1)}_{\rho x} = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\left(\vec{a}^{(0)}_{\nu\bar{x}} \times \vec{a}^{(0)}_{\rho\bar{x}}\right) - \frac{1}{2}\sum_{i=1}^{n} T_i^{\mu x\ \mu_1 x_1}\left(\vec{a}^{(0)}_{\mu_1 x_1} \times \vec{I}_i\right). \tag{2.92}$$

Taking the divergence of the equation (2.92), we have

$$0 = \vec{a}^{(0)}_{\mu\bar{x}} \times \left(\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}^{(0)}_{\rho\bar{x}}\right) - \frac{1}{2}\sum_{i=1}^{n}\partial_\mu T_i^{\mu x\ \mu_1 x_1}\left(\vec{a}^{(0)}_{\mu_1 x_1} \times \vec{I}_i\right), \tag{2.93}$$

since $\varepsilon^{\mu\nu\rho}\partial_\mu\partial_\nu\vec{a}^{(1)}_{\rho x} = 0$. Recall the general differential constraint (2.70), we can write it for the T-object with two indices as

$$\partial_\mu T_i^{\mu x\ \mu_1 x_1} = \left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_1 x_1}. \tag{2.94}$$

Replacing eqution (2.94) into (2.93), we obtain

$$\frac{1}{2}\sum_{j=1}^{n}\delta^3\left(x - x_j\right) T_j^{\mu_1 x_1}\left(\vec{a}^{(0)}_{\mu_1 x_1} \times \vec{I}_j\right) = 0. \tag{2.95}$$

The solution of the equation (2.88) can be written in terms of the metric (2.72) by using the equation (2.74) as follows

$$\vec{a}^{(0)}_{\mu x} = \frac{1}{2}\sum_i D_{i\ \mu x}\vec{I}_i(0) = -\frac{1}{2}\sum_i T_i^{\nu y} g_{\mu x\ \nu y}\vec{I}_i(0). \tag{2.96}$$

If we combine the equations (2.75), (2.96) and (2.95), the consistency condition for $\bar{\mu}(1,2,3)$ is given by

$$\frac{1}{4}\sum_{i,j}\delta^3\left(x - x_j\right)(I_i \times I_j)L(i,j) = 0. \tag{2.97}$$

In general, the function of the currents does not always vanish, so it holds that:

$$L(i,j) = 0, \tag{2.98}$$

since $S^{(1)}$ describes a linking invariant of three curves such that the GLN between each other is always zero as required.

2.7.3 $S^{(2)}$: Four-components link

Similarly, using the equation (2.47) we can write the second-order equation of motion as follows:

$$2\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}^{(2)}_{\rho x} = -2\varepsilon^{\mu\nu\rho}\vec{a}^{(1)}_{\nu\bar{x}}\times\vec{a}^{(0)}_{\rho\bar{x}} - \sum_{i=1}^{n} T_i^{\mu x\ \mu_1 x_1}\vec{a}^{(1)}_{\mu_1 x_1}\times\vec{I}_i + \sum_{i=1}^{n} T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{a}^{(0)}_{\mu_1 x_1}\times\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_i\right)\right], \tag{2.99}$$

If we take its divergence, we obtain

$$\begin{aligned} 0 = &-2\varepsilon^{\mu\nu\rho}\partial_\mu\vec{a}^{(1)}_{\nu\bar{x}}\times\vec{a}^{(0)}_{\rho\bar{x}} - 2\varepsilon^{\mu\nu\rho}\vec{a}^{(1)}_{\nu\bar{x}}\times\partial_\mu\vec{a}^{(0)}_{\rho\bar{x}} - \sum_i \partial_\mu T_i^{\mu\bar{x}\ \mu_1 x_1}\left[\vec{a}^{(1)}_{\mu_1 x_1}\times\vec{I}_i\right] \\ &+ \sum_i \partial_\mu T_i^{\mu\bar{x}\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{a}^{(0)}_{\mu_1 x_1}\times\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_i\right)\right], \end{aligned} \tag{2.100}$$

since $\varepsilon^{\mu\nu\rho}\partial_\mu\partial_\nu\vec{a}^{(2)}_{\rho x} = 0$. Now, by using the differential constraint (2.70), the equation of motion (2.92), and introducing the vector identities

$$\varepsilon^{\mu\nu\rho}\left(\vec{a}^{(0)}_{\mu\bar{x}}\times\vec{a}^{(0)}_{\nu\bar{x}}\right)\times\vec{a}^{(0)}_{\rho\bar{x}} = 0,$$

$$\left(\vec{a}^{(0)}_{\mu\bar{x}}\times\vec{a}^{(0)}_{\nu\bar{x}}\right)\times\vec{a}^{(0)}_{\rho\bar{x}} = \vec{a}^{(0)}_{\nu\bar{x}}\left(\vec{a}^{(0)}_{\mu\bar{x}}\cdot\vec{a}^{(0)}_{\rho\bar{x}}\right) - \vec{a}^{(0)}_{\mu\bar{x}}\left(\vec{a}^{(0)}_{\nu\bar{x}}\cdot\vec{a}^{(0)}_{\rho\bar{x}}\right),$$

in the equation (2.100), we can write the consistency condition as follows

$$\sum_i \delta^{(3)}\left(x - x_i(0)\right)\left[T_i^{\mu_1 x_1}\vec{a}^{(1)}_{\mu_1 x_1}\times\vec{I}_i - T_i^{\mu_1 x_1\ \mu_2 x_2}\vec{a}^{(0)}_{\mu_1 x_1}\times\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_i\right)\right] = 0. \tag{2.101}$$

Thus, from the previous expression, it can be concluded that what is within the brackets must vanish:

$$T_i^{\mu_1 x_1}\vec{a}^{(1)}_{\mu_1 x_1}\times\vec{I}_i - T_i^{\mu_1 x_1\ \mu_2 x_2}\vec{a}^{(0)}_{\mu_1 x_1}\times\left(\vec{a}^{(0)}_{\mu_2 x_2}\times\vec{I}_i\right) = 0. \tag{2.102}$$

If we replace (2.88) and the solution to the equation of motion (2.92) into (2.102), we have

$$\begin{aligned} &\sum_{j,k}\left[\left(\vec{I}_j\times\vec{I}_k\right)\times\vec{I}_i\right]\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho}T_i^a T_j^b T_k^c g_{\mu x\ a} g_{\nu x\ b} g_{\rho x\ c} + T_i^a T_j^{cd} T_k^b g_{ca} g_{db}\right\} \\ &-\sum_{j,k}\left[\vec{I}_j\times\left(\vec{I}_k\times\vec{I}_i\right)\right] T_j^a T_i^{cd} T_k^b g_{ca} g_{db} = 0. \end{aligned} \tag{2.103}$$

Here we used the Latin letter notation for discrete-continuous indices. At this moment, the last equation constitutes our most basic result for the consistency condition, which we will analyze immediately. Note that, if we take the dot product between the equation (2.103) and the vector $\vec{I}_i$, and using the following vector identities

$$\left[\left(\vec{I}_j \times \vec{I}_k\right) \times \vec{I}_i\right] \cdot \vec{I}_i = 0,$$

$$\left[\vec{I}_j \times \left(\vec{I}_k \times \vec{I}_i\right)\right] \cdot \vec{I}_i = \left(\vec{I}_i \cdot \vec{I}_j\right)\left(\vec{I}_i \cdot \vec{I}_k\right) - \left|\vec{I}_i\right|^2 \left(\vec{I}_j \cdot \vec{I}_k\right),$$

the consistency condition turns out to be

$$\sum_{j,k} \left[\left(\vec{I}_i \cdot \vec{I}_j\right)\left(\vec{I}_i \cdot \vec{I}_k\right) - \left|\vec{I}_i\right|^2 \left(\vec{I}_j \cdot \vec{I}_k\right)\right] T_j^a T_i^{cd} T_k^b g_{ca} g_{db} = 0, \tag{2.104}$$

which is symmetric between the indices j, k. Thus, by applying a symmetrization of the equation (2.104) and using the algebraic constraint (2.71) for the symmetric part of the T-objects $T^{(ab)} = T^a T^b$, we arrive at

$$\sum_{j,k} T_j^a T_i^c T_i^d T_k^b g_{ca} g_{db} = \sum_{j,k} \left(T_j^a g_{ca} T_i^c\right)\left(T_i^d g_{db} T_k^b\right) = \sum_{j,k} L(i,j) L(i,k) = 0, \tag{2.105}$$

so $L(i,j) = 0$ as required. Now, let us go back to the equation (2.103), where if we take the vector product between this equation and $\vec{I}_i$, and consider the following vector identities

$$\left[\left(\vec{I}_j \times \vec{I}_k\right) \times \vec{I}_i\right] \times \vec{I}_i = \left(\vec{I}_k \times \vec{I}_i\right)\left(\vec{I}_i \cdot \vec{I}_j\right) - \left(\vec{I}_j \times \vec{I}_i\right)\left(\vec{I}_i \cdot \vec{I}_k\right),$$

$$\left[\vec{I}_j \times \left(\vec{I}_k \times \vec{I}_i\right)\right] \times \vec{I}_i = \left(\vec{I}_k \times \vec{I}_i\right)\left(\vec{I}_i \cdot \vec{I}_j\right),$$

the consistency condition is now given by

$$\begin{aligned} &\sum_{i,j,k} \left[\left(\vec{I}_k \times \vec{I}_i\right)\left(\vec{I}_i \cdot \vec{I}_j\right)\right] \left\{\frac{1}{2}\varepsilon^{\mu\nu\rho} T_i^a T_j^b T_k^c g_{\mu x a} g_{\nu x b} g_{\rho x c} + \left(T_i^a T_j^{[cd]} T_k^b - T_j^a T_i^{[cd]} T_k^b\right) g_{ca} g_{db}\right\} \\ &- \sum_{i,j,k} \left[\left(\vec{I}_j \times \vec{I}_i\right)\left(\vec{I}_i \cdot \vec{I}_k\right)\right] \left\{\frac{1}{2}\varepsilon^{\mu\nu\rho} T_i^a T_j^b T_k^c g_{\mu x a} g_{\nu x b} g_{\rho x c} + T_i^a T_j^{[cd]} T_k^b g_{ca} g_{db}\right\} = 0. \end{aligned} \tag{2.106}$$

Here, we have appropriately antisymmetrized the term within braces, taking into account the antisymmetry present in the coefficients of the iso-charges ((i,k) in the first sum and (i,j) in the second). Interchanging the indices $(j \leftrightarrow k)$ in the second term of the last equation, and using (2.74) to write the consistency condition in terms of the D-objects, we have

$$\sum_{i,j,k} 2 f_{i,j,k} \bar{\mu}(i,j,k) = 0, \tag{2.107}$$

where $f_{i,j,k} = \left(\vec{I}_k \times \vec{I}_i\right)\left(\vec{I}_i \cdot \vec{I}_j\right)$ and

$$\bar{\mu}(i,j,k) = -\frac{1}{2}\Big[\varepsilon^{\mu\nu\rho} D_{i\ \mu x} D_{j\ \nu x} D_{k\ \rho x} + \left(T_i^{[\mu x\ \nu y]} D_{j\ \mu x} D_{k\ \nu y} + T_j^{[\mu x\ \nu y]} D_{i\ \nu y} D_{k\ \mu x} + T_k^{[\mu x\ \nu y]} D_{i\ \mu x} D_{j\ \nu y}\right)\Big]\,. \tag{2.108}$$

Note that as the function of the iso-currents $f_{i,j,k}$ in equation (2.107) is not generally zero, it implies that $\bar{\mu}(i,j,k) = 0$ as required. Therefore, $S^{(2)}$ describes four closed loops linked in the space-time (as we will show later), but not in the way of the GLN nor the third Milnor's coefficient.

Throughout this chapter, we derived the equations of motion for the Chern-Simons-Wong theory. Subsequently, we stated that the fields $A^a_\mu = A^a_\mu[\gamma_i]$ are functionals of the, already given, closed trajectories of Wong particles in space-time. By substituting the solutions of the equations of motion into the action, we obtained the on-shell action, which depends solely on closed curves (all dependence on the A^a_μ fields disappears). Due to this, and the topological nature of the action, the only topological invariant that can be described by an object depending on closed curves is the way in which they are linked. Therefore, $S_{on-shell}$ must be a link invariant. As analyzing the complete on-shell action proves to be a nontrivial task, a perturbative analysis was developed, in which we obtained link invariants at different orders. As this procedure is somewhat tedious, then the idea that there might exist some intermediate Abelian theory (a cousin of the Abelian and non-Abelian Chern-Simons theories), which could describe a higher-order link invariant arises naturally. This idea will be explored thoroughly in the next chapter.

Chapter 3

Intermediate Theories

In Chapter 1, we saw how the Abelian Chern-Simons theory coupled with a suitable current reproduced the GLN. On the other hand, in chapter 2, we derived various knot invariants through a perturbative analysis of the non-Abelian Chern-Simons theory coupled with Wong particles. Since the last method is highly nontrivial, one could ask if there exist an intermediate Abelian theory that is capable of reproducing higher-order knot invariants, such as those obtained from the non-Abelian Chern-Simons theory. Stated more precisely: "is there any topological field theory, other than the Abelian Chern-Simons theory that yields exact analytical expressions for link invariants, other than the GLN?". This idea will be explored throughout this chapter. The first section is based on the work of Leal and Pineda [3], which consists of an intermediate Abelian theory that exactly reproduces the TMC. Meanwhile, the second section encompasses the original development of an intermediate Abelian theory that describes a four-components link.

3.1 Intermediate action for the third Milnor's coefficient $\mu(1,2,3)$

The intermediate Abelian action, proposed in [3], is given by

$$\begin{aligned}\tilde{S} = &\int d^3x \varepsilon^{\mu\nu\rho} \left\{ 4A^i_\mu(x)\partial_\nu a_{i\rho}(x) + \frac{2}{3}\varepsilon^{ijk} a_{i\mu}(x) a_{j\nu}(x) a_{k\rho}(x) \right\} \\ &- 2\int d^3x T^{\mu x}_i A^i_\mu(x) + \int d^3x \int d^3y \varepsilon^{ijk} T^{\mu x,\nu y}_i a_{j\mu}(x) a_{k\nu}(y),\end{aligned} \tag{3.1}$$

where $A^\mu_i(x)$ and $a^\mu_i(x)$ represent two independent sets of Abelian gauge fields, denoted by Latin letters running from 1 to 3 (we are using the Einstein summation convention for these "internal" indices. It is important to note that we do not employ its generalization since the integration of continuous variables appears explicitly). The first two terms correspond to the topological theory with a non-semisimple gauge group of symmetry as introduced in reference [8].

Performing variations of the action ($\delta\tilde{S}=0$) with respect to the fields ($A^i_{\mu x}$, $a^i_{\mu x}$) yields

$$\varepsilon^{\mu\nu\rho}\partial_\nu a_{i\rho} = \frac{1}{2}T_i^{\mu x}, \tag{3.2}$$

$$\varepsilon^{\mu\nu\rho}\partial_\nu A^i_\rho(x) = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{ijk}a_{j\nu}(x)a_{k\rho}(x) + \frac{1}{2}\int d^3y\varepsilon^{ijk}T_j^{[\mu x,\nu y]}a_{k\nu}(y). \tag{3.3}$$

These equations are just the zeroth and first-order contributions to the $SU(2)$ Chern-Simons-Wong equations of motion that were studied in chapter 2.

By taking the divergence of the equation (3.2), we get

$$\partial_\mu T_i^{\mu x} = 0, \tag{3.4}$$

since $\varepsilon^{\mu\nu\rho}\partial_\mu\partial_\nu a_{i\rho} = 0$. This reflects the gauge invariance of the action under the transformations

$$A^i_\mu \longrightarrow A^i_\mu + \partial_\mu \Lambda^i, \tag{3.5}$$

where $\Lambda^i = \Lambda^i(x)$ is an arbitrary function. The consistency condition for the equation (3.3) involves more intricate calculations. If we take its divergence, we have

$$0 = 2\varepsilon^{\mu\nu\rho}\varepsilon^{ijk}\partial_\mu\left(a_{j\nu}(x)a_{k\rho}(x)\right) - \varepsilon^{ijk}\int d^3y a_{k\nu}(y)\frac{\partial}{\partial x^\mu}T_j^{[\mu x,\nu y]}. \tag{3.6}$$

Using the differential constraint (2.70), and the equation of motion (3.2) for the field $a^i_\mu(x)$ into the equation (3.6), it follows

$$\varepsilon^{ijk}\delta^3\left(x - x_j(0)\right)\oint_j dx^\nu \oint_k dz^\beta \varepsilon_{\nu\alpha\beta}\frac{(x-z)^\alpha}{|x-z|^3} = \varepsilon^{ijk}\delta^3\left(x-x_j(0)\right)L(j,k) = 0, \tag{3.7}$$

so the GLN must vanish as required, $L(i,j)=0, \forall i,j$. From this result, we deduce that the theory maintains consistency as long as the curves are not linked according to the GLN. However, this does not imply that the curves are equivalent to the trivial link (the unlink). For example, the Borromean Rings are a well-known set of three curves with vanishing GLN between them, even though they are entangled.

The consistency condition (3.7) is also related to a gauge symmetry of the theory. A direct calculation shows that the action (3.1) is invariant under the transformations

$$a_{i\mu} \to a_{i\mu} + \partial_\mu \Omega_i, \tag{3.8}$$

provided that the consistency condition (3.7) is fulfilled. Note that $\Omega_i = \Omega_i(x)$ is an arbitrary function. Hence, it is evident that both sets of fields A_i and a_i must be Abelian gauge fields for the theory to maintain consistency.

There is no need to introduce a metric on the manifold to construct the action, as can be easily verified. Therefore, the theory is metric-independent. Given its general covariance, it qualifies as a topological field theory, just like its cousins the Abelian and non-Abelian Chern-Simons theories. Consequently, we infer that the on-shell action $\tilde{S}_{on-shell}$ of the theory should exclusively depend on the topological features of the curves involved in the action; in other words, it should be a link invariant. Let us examine how this unfolds. The solution to the equation of motion (3.2) for the field $a^i_\mu(x)$ is given by

$$a_{i\mu}(x) = -\left(\frac{1}{2}\right)\frac{1}{4\pi}\oint_{\gamma_i} dz^\rho \varepsilon_{\mu\nu\rho}\frac{(x-z)^\nu}{|x-z|^3}. \tag{3.9}$$

Note that the equation (3.3) can also be integrated as easily as the former one, but in order to calculate $\tilde{S}_{on-shell}$ it suffices to substitute the left hand side of (3.3) and the equation (3.9) into (3.1). Thus, the on-shell action becomes

$$\begin{aligned}\tilde{S}(1,2,3) = &-\frac{1}{2}\int d^3x \varepsilon^{\mu\nu\rho} a_{1\mu}(x)a_{2\nu}(x)a_{3\rho}(x)\\ &-\frac{1}{2}\int d^3x\int d^3y\,\Big(T_1^{[\mu x,\nu y]}a_{2\mu}(x)a_{3\nu}(y)+\\ &+T_2^{[\mu x,\nu y]}a_{3\mu}(x)a_{1\nu}(y)+T_3^{[\mu x,\nu y]}a_{1\mu}(x)a_{2\nu}(y)\Big).\end{aligned} \tag{3.10}$$

Equation (3.10) corresponds to an analytical expression for Milnor's Linking Coefficient $\bar{\mu}(1,2,3)$, and it corresponds exactly to equation (2.61) obtained in chapter 2. So, an intermediate Abelian theory has been developed to describe a higher-order link, such as the TMC. Now, following this intermediate action methodology, in the next section we will develop an Abelian theory that describes the four-components link corresponding to the on-shell action $S^{(2)}$ of the non-Abelian Chern-Simons-Wong theory.

3.2 Intermediate action for the four-components link

As an intermediate Abelian action for the four-components link we propose

$$
\begin{aligned}
S_M = & -6\int d^3x\ \vec{\Lambda}_{\mu x}\cdot\left[\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\lambda}_{\rho x}-\frac{1}{2}\sum_i^n T_i^{\mu x}\vec{I}_i\right] \\
& +\frac{3}{4}\int d^3x\int d^3y\ \varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\mu x\ \alpha y}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\cdot\vec{\lambda}_{\rho x} \\
& +\frac{3}{2}\sum_{i=1}^n\int d^3x\int d^3y\int d^3x_1\ \varepsilon^{\mu\nu\rho}g_{\mu x\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\vec{\lambda}_{\rho x} \\
& +\frac{3}{4}\sum_{i,j}\int d^3x\int d^3y\int d^3x_1\int d^3x_2\ g_{\mu x\ \alpha y}T_i^{[\mu_1 x_1\ \mu x]}T_j^{[\alpha y\ \mu_2 x_2]}\left[\vec{I}_i\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\right]\cdot\vec{\lambda}_{\mu_1 x_1} \\
& +\sum_{i=1}^n\int d^3x\int d^3x_1\int d^3x_2\ T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\vec{\lambda}_{\mu x},
\end{aligned}
\tag{3.11}
$$

where $\vec{\lambda}_{\mu x}$ and $\vec{\Lambda}_{\mu x}$ are two sets of independent Abelian gauge fields, labeled by Latin letters running from 1 to 3, as well as the current $\vec{I}_i$, corresponding to the i-th particle. Here we are using vector notation for internal indices, i.e., the fields can be seen as $(\lambda^a_{\mu x},\ \Lambda^a_{\mu x})$, and the current as I^a_i. These objects are "living" in the internal space.

Now, using the notation for "discrete-continuous" indices, the intermediate action for the four-components link is written in a contracted way as

$$
\begin{aligned}
S_M = & -6\vec{\Lambda}_{\mu x}\cdot\left[\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\lambda}_{\rho x}-\frac{1}{2}\sum_i^n T_i^{\mu x}\vec{I}_i\right] \\
& +\frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\mu x\ \alpha y}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\cdot\vec{\lambda}_{\rho x} \\
& +\frac{3}{2}\sum_{i=1}^n\varepsilon^{\mu\nu\rho}g_{\mu x\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\vec{\lambda}_{\rho x} \\
& +\frac{3}{4}\sum_{i,j}g_{\mu x\ \alpha y}T_i^{[\mu_1 x_1\ \mu x]}T_j^{[\alpha y\ \mu_2 x_2]}\left[\vec{I}_i\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\right]\cdot\vec{\lambda}_{\mu_1 x_1} \\
& +\sum_{i=1}^n T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\vec{\lambda}_{\mu x}.
\end{aligned}
\tag{3.12}
$$

If we perform variations of the action ($\delta S_M = 0$) with respect to the field $\vec{\Lambda}_{\mu x}$, we obtain:

$$
\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\lambda}_{\rho x}=\frac{1}{2}\sum_i^n T_i^{\mu x}\vec{I}_i, \tag{3.13}
$$

whose solution is given by:

$$\vec{\lambda}_{\mu x} = \frac{1}{2}\sum_i D_{i\ \mu x}\vec{I}_i = -\frac{1}{2}\sum_i g_{\mu x\ \nu y}T_i^{\nu y}\vec{I}_i = -\frac{1}{2}\sum_i \frac{1}{4\pi}\oint_{\gamma_i} dz^\gamma \frac{(x-z)^\beta}{|x-z|^3}\varepsilon_{\mu\beta\gamma}\vec{I}_i. \tag{3.14}$$

Variations with respect to $\vec{\lambda}$ leads to:

$$\begin{aligned} 2\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\Lambda}_{\rho x} = &- \varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\rho\bar{x}\ \alpha y}\left[\vec{\lambda}_{\nu\bar{x}} \times \left(\vec{\lambda}_{\beta y} \times \vec{\lambda}_{\gamma y}\right)\right] \\ &- \sum_{i=1}^n \varepsilon^{\mu\nu\rho}g_{\rho\bar{x}\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu\bar{x}} \times \left(\vec{\lambda}_{\mu_1 x_1} \times \vec{I}_i\right)\right] \\ &- \frac{1}{2}\sum_{i=1}^n \varepsilon^{\alpha\beta\gamma}g_{\mu_1 x_1\ \alpha y}T_i^{[\mu x\ \mu_1 x_1]}\left[\left(\vec{\lambda}_{\beta y} \times \vec{\lambda}_{\gamma y}\right) \times \vec{I}_i\right] \\ &- \frac{1}{2}\sum_{i,j} g_{\mu_1 x_1\ \alpha y}T_i^{[\mu x\ \mu_1 x_1]}T_j^{[\alpha y\ \mu_2 x_2]}\left[\left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_j\right) \times \vec{I}_i\right] \\ &+ \sum_{i=1}^n T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right], \end{aligned} \tag{3.15}$$

where

$$T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle} = \frac{2}{3}\left(T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2} - T_i^{[\mu_1 x_1\ \mu_2 x_2]\ \mu x}\right), \tag{3.16}$$

$$T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2} = \frac{1}{2}\left(T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2} - T_i^{\mu_1 x_1\ \mu x\ \mu_2 x_2}\right). \tag{3.17}$$

Equation (3.15) can be rewritten as

$$\begin{aligned} 2\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\Lambda}_{\rho x} = &- 2\varepsilon^{\mu\nu\rho}\vec{a}_{\nu\bar{x}} \times \vec{\lambda}_{\rho\bar{x}} - \sum_{i=1}^n T_i^{[\mu x\ \mu_1 x_1]}\vec{a}_{\mu_1 x_1} \times \vec{I}_i \\ &+ \sum_{i=1}^n T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right], \end{aligned} \tag{3.18}$$

similar to the second-order equation of motion for the non-Abelian theory, where

$$\vec{a}_{\mu x} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma}\left(\vec{\lambda}_{\beta y} \times \vec{\lambda}_{\gamma y}\right)g_{\mu x\ \alpha y} + \frac{1}{2}\sum_{i=1}^n T_i^{[\alpha y\ \mu_1 x_1]}\left(\vec{\lambda}_{\mu_1 x_1} \times \vec{I}_i\right)g_{\mu x\ \alpha y}. \tag{3.19}$$

Contracting the previous equation with the inverse metric tensor and defining $I_{lk} = \delta_{lk}$, we have

$$\varepsilon^{\mu\nu\rho}\partial_\nu a_{i\rho}(x) = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{ijk}\lambda_{j\nu}(x)\lambda_{k\rho}(x) + \frac{1}{2}\int d^3y\varepsilon^{ijk}T_j^{[\mu x\ \nu y]}\lambda_{k\nu}(y), \tag{3.20}$$

which is exactly the equation of motion for intermediate theory shown in the last section and developed in [3]. Substituting (3.13) and (3.14) into (3.12), the on-shell action is written as

$$\begin{aligned}S_{M\ on-shell} = \ & \frac{1}{8}\sum_{i,j,k,l}\left(\left(\vec{I}_i \times \left(\vec{I}_j \times \vec{I}_k\right)\right)\cdot \vec{I}_l\right) g_{\mu x\ \alpha y} \\ & \left\{\frac{3}{8}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\nu x\ \mu_1 x_1} g_{\beta y\ \mu_2 x_2} g_{\gamma y\ \mu_3 x_3} g_{\rho x\ \mu_4 x_4} T_i^{\mu_1 x_1} T_j^{\mu_2 x_2} T_k^{\mu_3 x_3} T_l^{\mu_4 x_4}\right. \\ & + \frac{3}{2}\varepsilon^{\mu\nu\rho} T_j^{[\alpha y\ \mu_1 x_1]} g_{\nu x\ \mu_2 x_2} g_{\mu_1 x_1\ \mu_3 x_3} g_{\rho x\ \mu_4 x_4} T_i^{\mu_2 x_2} T_k^{\mu_3 x_3} T_l^{\mu_4 x_4} \\ & + \frac{3}{2} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]} g_{\mu_2 x_2\ \mu_3 x_3} g_{\mu_1 x_1\ \mu_4 x_4} T_k^{\mu_3 x_3} T_l^{\mu_4 x_4} \\ & \left. + T_j^{\mu x\ \mu_1 x_1\ \mu_2 x_2} g_{\mu_1 x_1\ \mu_3 x_3} g_{\mu_2 x_2\ \mu_4 x_4} T_i^{\mu_3 x_3} T_k^{\mu_4 x_4} T_l^{\alpha y}\right\},\end{aligned} \tag{3.21}$$

which coincides with (2.90), as expected.

As discussed previously, both the GLN and the TMC must be zero for consistency in order to have a well defined four-components link invariant. Indeed, this is what occurs here and we will demonstrate this in what follows. Taking the divergence of equation (3.13), we arrive at

$$\partial_\mu T_i^{\mu x} = 0, \tag{3.22}$$

which coincides with the differential constraint of the equation (2.70). Taking the divergence of (3.20)

$$\frac{1}{4}\sum_{i,j}\delta^3\left(x - x_j\right)\left(I_i \times I_j\right) L(i,j) = 0, \tag{3.23}$$

where $L(i,j)$ is the GLN, as defined in (2.75). Then, taking the divergence of (3.15), we arrive at

$$\sum_{i,j,k} 2 f_{i,j,k}\bar{\mu}(i,j,k) = 0, \tag{3.24}$$

where $f_{i,j,k} = \left(\vec{I}_k \times \vec{I}_i\right)\left(\vec{I}_i \cdot \vec{I}_j\right)$ and

$$\begin{aligned}\bar{\mu}(i,j,k) = & -\frac{1}{2}\Big[\epsilon^{\mu\nu\rho} D_{i\ \mu x} D_{j\ \nu x} D_{k\ \rho x} \\ & + \left(T_i^{[\mu x\ \nu y]} D_{j\ \mu x} D_{k\ \nu y} + T_j^{[\mu x\ \nu y]} D_{i\ \nu y} D_{k\ \mu x} + T_k^{[\mu x\ \nu y]} D_{i\ \mu x} D_{j\ \nu y}\right)\Big],\end{aligned} \tag{3.25}$$

with $D_{i\ a} = D_{i\ \mu x}$ as in equation (2.74). Note that (3.25) corresponds to the TMC, as demonstrated in [2]. It is worth emphasizing that equations (3.23) and (3.24) imply that $L(i,j) = 0$ and $\bar{\mu}(i,j,k) = 0$, respectively, as required. Thus, we have constructed an intermediate Abelian theory, such that describes four closed curves linked in space-time, but not in the way of the GLN nor the TMC, and it corresponds exactly to the expression for $S^{(2)}$ that we constructed in chapter 2. If you wish to see the calculations concerning to this section in more detail, you can refer to Appendix A.

Throughout this work, we have calculated, through two different methods (perturbative analysis and intermediate action methodology) the analytical expressions for the following link invariants: GLN, TMC, and the four-components link. Now, we must verify that these expressions are indeed related to these link invariants. To do so, two forms of geometric interpretation will be introduced in the next chapter, providing a clear demonstration of this relationship.

Chapter 4

Geometrical Meaning

In this chapter, we will demonstrate that the analytical expressions (2.53), (2.61), and (2.90) detect the link invariants related to the GLN, the TMC, and the four-component link, respectively. To achieve this, we will introduce two ways of geometric interpretation. The first one was developed in [5], and involves drawing open paths diagrams (coming from infinity and ending at the edges of the closed curves under study), which are related to the Seifert surfaces of these curves. Thus, it is possible to determine the required knot invariant by observing the relationships that the tangent vectors of the curves and the open paths have with each other. The second method involves rewriting these analytical expressions as surface integrals and determining the relationships that the normal vectors to the surfaces enclosed by these curves satisfy.

4.1 Open paths interpretation

4.1.1 Gauss linking number

First, we start with the simplest of the three link invariants we have obtained, the GLN. All the tools developed in this subsection will be very useful for interpreting higher-order link invariants. Consider the kernel of the GLN, which is given by the following expression:

$$k^{\rho}(x,z) = \frac{1}{4\pi}\frac{(x-z)^{\rho}}{|x-z|^{3}} = -\frac{1}{4\pi}\partial^{\rho}\left(\frac{1}{|x-z|}\right). \tag{4.1}$$

Taking its divergence, note that

$$\partial^{x}_{\rho}k^{\rho} = -\frac{1}{4\pi}\nabla^{2}\left(\frac{1}{|x-z|}\right) = \delta^{(3)}(x-z). \tag{4.2}$$

We define the form factor for an open curve γ^z, coming from infinity and ending at the point z, as follows

$$h^\rho(x,\gamma^z) = \int_{\gamma^z} dy^\rho \delta^{(3)}(x-y). \tag{4.3}$$

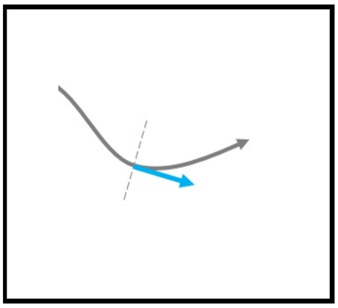

Figure 4.1: Tangent vector of a curve at a point.

The form factor (4.3) can be interpreted as an object that takes the tangent vector of a curve at a given point, as shown Figure 4.1. Taking its divergence, we have

$$\partial_\rho h^\rho = \int_{\gamma^z} dy^\rho \partial_\rho(\delta^{(3)}(x-y)) = \delta^{(3)}(x-z). \tag{4.4}$$

Since objects k^ρ and h^ρ have the same divergence, then they differ by the curl of a vector field. This allows us to make the change $k^\rho \;\; \rightarrow \;\; h^\rho$. So, we can perform a gauge transformation on the object $h^\rho(x)$, where the open path γ^z is parallel transported along the closed curve to which it ends, creating a diagram of open strands parallel to each other (analogous to Seifert surfaces). Therefore, the GLN could be written as

$$L(i,j) = \oint_{\gamma_i} dx^\mu \oint_{\gamma_j} dy^\nu \varepsilon_{\mu\nu\rho} \int_{\gamma^y} dz^\rho \delta^{(3)}(x-z). \tag{4.5}$$

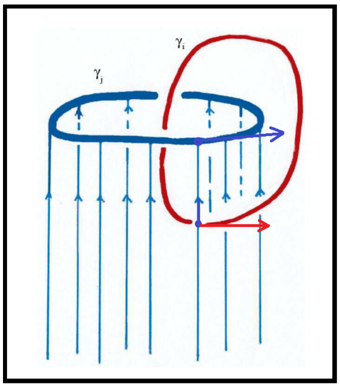

Figure 4.2: Open strands diagram for the GLN.

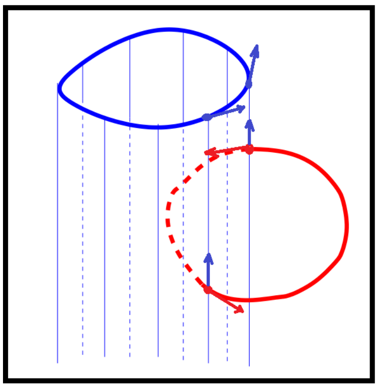

Figure 4.3: Trivial Link (not linked according to the open strands diagram).

The expression in (4.5) tells us that the mixed product between the tangent vectors to the open paths γ^y and the closed curves γ_j (blue) and γ_i (red) must be non-degenerate, as depicted in Figure 4.2, with GLN equal to one. Otherwise, there is no linking between the curves, and we obtain the trivial link as shown in Figure 4.3. Similarly, employing the tools developed thus far, we will interpret geometrically the analytical expression for the TMC in the following subsection.

4.1.2 Borromean rings

Using the conventions and notation developed in section 2.5, it is possible to write the expression (2.61) for $S^{(1)}$ in terms of the metric and the T-objects as follows

$$
\begin{aligned}
S^{(1)} = & -\frac{1}{2}\sum_{i,j,k}\left[\left(\vec{I}_i\times\vec{I}_j\right)\cdot\vec{I}_k\right]\Big[\varepsilon^{\mu\nu\rho}T_i^{\mu_1 x_1}T_j^{\mu_2 x_2}T_k^{\mu_3 x_3}g_{\mu x\ \mu_1 x_1}g_{\nu x\ \mu_2 x_2}g_{\rho x\ \mu_3 x_3} \\
& + T_i^{[\mu x\ \nu y]}T_j^{\mu_1 x_1}T_k^{\mu_2 x_2}g_{\mu x\ \mu_1 x_1}g_{\nu y\ \mu_2 x_2} + T_j^{[\mu x\ \nu y]}T_k^{\mu_1 x_1}T_i^{\mu_2 x_2}g_{\mu x\ \mu_1 x_1}g_{\nu y\ \mu_2 x_2} \\
& + T_k^{[\mu x\ \nu y]}T_i^{\mu_1 x_1}T_j^{\mu_2 x_2}g_{\mu x\ \mu_1 x_1}g_{\nu y\ \mu_2 x_2]}\Big]\,.
\end{aligned}
\tag{4.6}
$$

Let us define the object $H_{\mu\nu}(x,\gamma)$ for general open curves

$$
H_{\mu\nu}(x,\gamma)\equiv\int_\gamma dy^\rho\varepsilon_{\mu\nu\rho}\delta^{(3)}(x-y), \tag{4.7}
$$

which could be related to the form factor as $H_{\mu\nu}(x,\gamma^z)=\varepsilon_{\mu\nu\lambda}h^\lambda(x,\gamma^z)$. Also, the metric in (2.72) could be written in terms of the kernel for the GLN as $g_{\mu x\ \nu y}=-\varepsilon_{\mu\nu\rho}k^\rho(x,z)$. By the same argument in the last section, since h^ρ and k^ρ differ by the curl of a vector field, we can change $-g_{\mu x\ \nu y}\to H_{\mu\nu}\left(x,\gamma^{\vec{y}}\right)$. Thus, the first term of the equation (4.6) can be written in terms of $H_{\mu\nu}$ as follows

$$
\begin{aligned}
& \varepsilon^{\mu\nu\rho}T_i^{\mu_1 x_1}T_j^{\mu_2 x_2}T_k^{\mu_3 x_3}g_{\mu x\ \mu_1 x_1}g_{\nu x\ \mu_2 x_2}g_{\rho x\ \mu_3 x_3}\longrightarrow \\
& -\varepsilon^{\mu\nu\rho}\int d^3x\oint_{\gamma_i}dz_1^\alpha\oint_{\gamma_j}dz_2^\beta\oint_{\gamma_k}dz_3^\gamma H_{\mu\alpha}\left(x,\gamma^{\vec{z_1}}\right)H_{\nu\beta}\left(x,\gamma^{\vec{z_2}}\right)H_{\rho\gamma}\left(x,\gamma^{\vec{z_3}}\right) \\
& = -\varepsilon^{\mu\nu\rho}\oint_{\gamma_i}dz_1^\alpha\oint_{\gamma_j}dz_2^\beta\oint_{\gamma_k}dz_3^\gamma\int_{\gamma^{\vec{z_1}}}dy_1^\delta\int_{\gamma^{\vec{z_2}}}dy_2^\lambda\int_{\gamma^{\vec{3}}3}dy_3^\sigma\varepsilon_{\mu\alpha\delta}\varepsilon_{\nu\beta\lambda}\varepsilon_{\rho\gamma\sigma}\quad\times \\
& \times\quad\delta^{(3)}\left(y_2-y_1\right)\delta^{(3)}\left(y_3-y_1\right).
\end{aligned}
\tag{4.8}
$$

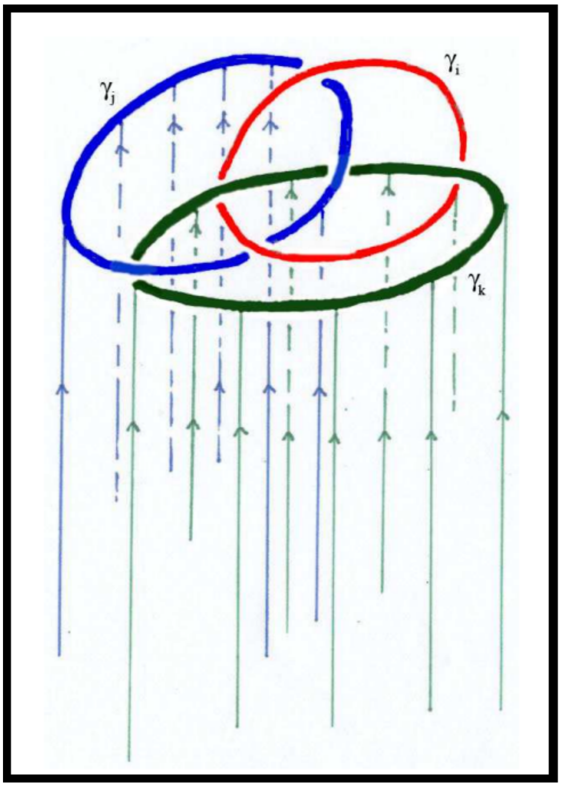

Figure 4.4: Open paths diagram for the Borromean rings.

Equation (4.8) indicates the number of times the surfaces generated by the open paths $\gamma^{\vec{z}_1}$, $\gamma^{\vec{z}_2}$, and $\gamma^{\vec{z}_3}$, which end on the closed trajectories γ_i, γ_j, and γ_k respectively of the particles, intersect at the same point. Now, if we consider (as we are doing) that the bundles associated with all closed trajectories of the particles are parallel, this term will be always zero, as shown in Figure 4.4. The second term of the equation (4.6) can be written in terms of $H_{\mu\nu}$ as follows

$$\begin{aligned} & T_i^{\mu x\ \nu y} T_j^{\mu_1 x_1} T_k^{\mu_2 x_2} g_{\mu x\ \mu_1 x_1} g_{\nu y\ \mu_2 x_2} \longrightarrow \\ & \oint_{\gamma_i} dz_1^\mu \int_0^{z_1} dz_2^\nu \oint_{\gamma_j} dz_3^\alpha \oint_{\gamma_k} dz_4^\beta H_{\mu\alpha}\left(z_1, \gamma^{\overrightarrow{z_3}}\right) H_{\nu\beta}\left(z_2, \gamma^{\overrightarrow{z_4}}\right) \\ & = \oint_{\gamma_i} dz_1^\mu \int_0^{z_1} dz_2^\nu \oint_{\gamma_j} dz_3^\alpha \oint_{\gamma_k} dz_4^\beta \varepsilon_{\mu\alpha\rho} \int_{\gamma^{\overrightarrow{z_3}}} dy_1^\rho \delta^{(3)}\left(y_1 - z_1\right) \varepsilon_{\nu\beta\lambda} \int_{\gamma^{\overrightarrow{z_4}}} dy_2^\lambda \delta^{(3)}\left(y_2 - z_2\right). \end{aligned} \tag{4.9}$$

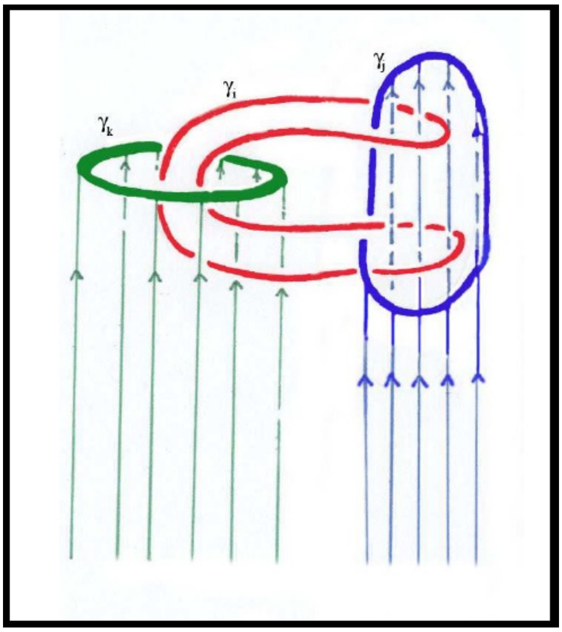

Figure 4.5: Open paths diagram for a deformation of the Borromean rings.

Note that we use the complete T-object and not just its anti-symmetric part to simplify the analysis. Equation (4.9) measures the oriented number of times a trajectory (for example, γ_i) intersects first the open paths attached to the second closed trajectory (such as γ_k) and then the open paths $\gamma^{\overrightarrow{z_3}}$ terminating on the third trajectory (for example, γ_j). Such an example can be seen in Figure 4.5. Additionally, each intersection between the tangent vectors of the particle trajectories dz and the tangent vectors of the open paths dy is nonzero only if the vectors form a non-degenerate volume, and the orientation of the curves fixes the direction of the tangent vectors, allowing us to distinguish the intersections as either entry or exit points on the surface generated by the "rain" of open paths. Note that there is no need to analyze the remaining terms in equation (4.6), as the terms following the second one are permutations of the indices (i, j, k) that label the particles.

We can always assume that the tangent vectors to the parallel open paths are in a fixed direction, for example, the 3-direction. In this case, each time a tangent vector from these open paths dy^α appears, it would indicate a vector in the dy^3 direction. It is clear that we can think of the set of parallel open paths terminating on a closed trajectory as a particular Seifert surface (a "kind" of cylinder without a top). Now, we are ready to provide a geometric interpretation of the second order on-shell action $S^{(2)}$ and show that it is associated with the 4-component link.

4.1.3 Four-components link

Just like in the previous subsection, we will change the metric g by the H-object as follows $-g_{\mu x\ \nu y} \to H_{\mu\nu}\left(x, \gamma^{\vec{y}}\right)$. Thus, the third term of the equation(2.90) is given by

$$
\begin{aligned}
&\frac{3}{2} T_j^{b\ \alpha y} T_i^{a\ \mu x} g_{bc} g_{ad} T_k^c T_l^d g_{\mu x\ \alpha y} \longrightarrow \\
&\frac{3}{2} \oint_{\gamma_i} dz_1^{\mu_1} \int_0^{z_1} dz_2^{\mu_2} \oint_{\gamma_j} dz_3^{\mu_3} \int_0^{z_3} dz_4^{\mu_4} \oint_{\gamma_k} dz_5^{\mu_5} \oint_{\gamma_l} dz_6^{\mu_6} H_{\mu_3\mu_5}\left(z_3, \gamma^{\vec{z_5}}\right) H_{\mu_1\mu_6}\left(z_1, \gamma^{\vec{b_6}}\right) H_{\mu_2\mu_4}\left(z_2, \gamma^{\vec{z_4}}\right) \\
&= \frac{3}{2} \oint_{\gamma_i} dz_1^{\mu_1} \int_0^{z_1} dz_2^{\mu_2} \oint_{\gamma_j} dz_3^{\mu_3} \int_0^{z_3} dz_4^{\mu_4} \oint_{\gamma_k} dz_5^{\mu_5} \oint_{\gamma_l} dz_6^{\mu_6} \varepsilon_{\mu_3\mu_5\alpha_1} \int_{\gamma^{z_5}} dy_1^{\alpha_1} \delta^{(3)}\left(y_1 - z_3\right) \times \\
&\quad \times \varepsilon_{\mu_1\mu_6\alpha_2} \int_{\gamma^{z_6}} dy_2^{\alpha_2} \delta^{(3)}\left(y_2 - z_1\right) \varepsilon_{\mu_2\mu_4\alpha_3} \int_{\gamma^{\vec{z}_4}} dy_3^{\alpha_3} \delta^{(3)}\left(y_3 - z_2\right).
\end{aligned}
\tag{4.10}
$$

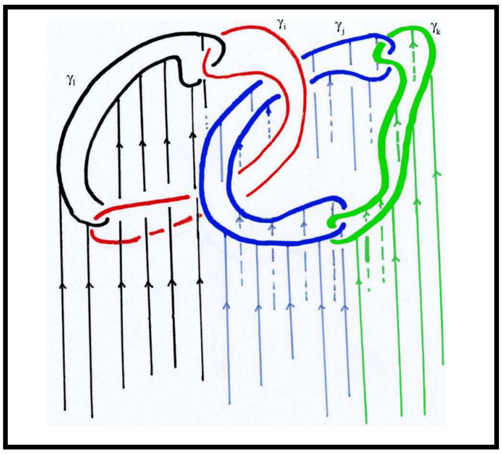

Figure 4.6: Open strands diagram for the 4-component link.

From this expression, we can argue that to have a contribution, it must happen that the closed trajectory γ_i crosses orderly first the open paths $\gamma_{\vec{z}_4}$ ending on the curve γ_j. Then, it must intersect at some point with the open paths belonging to $\gamma_{\vec{z}_6}$ terminating on the curve associated with particle l. Also, to demonstrate a non-zero contribution, we must have the curve γ_j intersecting at some point with the open paths $\gamma_{\vec{z}_5}$ associated with the trajectory γ_k, as shown in Figure 4.6.

Similarly, it is possible to determine the contributions of the other terms of the analytical expression for the four-components link, but since this method is somewhat cumbersome when analyzing higher-order link invariants, the remaining terms will be examined using the new geometric interpretation to be developed in the following section.

4.2 Surface integral interpretation

4.2.1 Gauss linking number

Throughout this section, we will develop a new method of geometric interpretation based on surface integrals. Similar to what was done in the previous section, we will start by constructing it from the GLN,

$$\begin{aligned} L(i,j) &= \frac{1}{4\pi}\oint_{\gamma_i} dx^\mu \oint_{\gamma_j} dz^b \varepsilon_{\mu bc}\frac{(x-z)^c}{|x-z|^3} \\ &= -\frac{1}{4\pi}\oint_{\gamma_i} dx^\mu \oint_{\gamma_j} dz^b \varepsilon_{\mu bc}\partial^c\left(\frac{1}{|x-z|}\right). \end{aligned} \tag{4.11}$$

We want to express the internal closed path integral as a surface integral. For this, we need Stokes' theorem

$$\oint_\gamma \vec{F}\cdot d\vec{l} = \int_\phi d\vec{\Sigma}\cdot(\vec{\nabla}\times\vec{x}) \longrightarrow \oint_\gamma F^a dx^a = \int_\delta d\Sigma^a \varepsilon^{abc}\partial^b F^c. \tag{4.12}$$

Replacing the last equation into (4.11), we can write the GLN as

$$\begin{aligned} L(i,j) &= -\frac{1}{4\pi}\oint_{\gamma_i} dx^\mu \int_{S[\gamma_j,\vec{y}]} d\Sigma_d \varepsilon^{deb}\partial_e\left(\varepsilon_{\mu bc}\partial^c\left[\frac{1}{|\vec{x}-\vec{y}|}\right]\right) \\ &= \frac{1}{4\pi}\oint_{\gamma_j} dx^\mu \int_{S[\gamma_i,\vec{y}]} d\Sigma_d \varepsilon^{bde}\varepsilon_{b\mu c}\partial_e\left(\partial^c\left[\frac{1}{|\vec{x}-\vec{y}|}\right]\right) \\ &= \frac{1}{4\pi}\oint_{\gamma_i} dx^\mu \int_{S[\gamma_j,\vec{y}]} d\Sigma_d (\delta^d_\mu\delta^e_c - \delta^d_c\delta^e_\mu)\partial_e\left(\partial^c\left[\frac{1}{|\vec{x}-\vec{y}|}\right]\right) \\ &= \frac{1}{4\pi}\oint_{\gamma_i} dx^\mu \int_{S[\gamma_j,\vec{y}]} d\Sigma_\mu \nabla^2\left[\frac{1}{|\vec{x}-\vec{y}|}\right] \\ &\quad - \frac{1}{4\pi}\oint_{\gamma_i} dx^\mu \int_{S[\gamma_j,\vec{y}]} d\Sigma_c \partial_\mu\left(\partial^c\left[\frac{1}{|\vec{x}-\vec{y}|}\right]\right). \end{aligned} \tag{4.13}$$

Note that the last term in the above equation must vanish, since we have a line integral over a closed curve of a total differential,

$$L(i,j) = \oint_{\gamma_i} dx^\mu \int_{S[\gamma_j,\vec{y}]} d\Sigma_\mu \delta^3(\vec{x}-\vec{y}) \\ - \frac{1}{4\pi}\int_{S[\gamma_j,\vec{y}]} d\Sigma_c \underbrace{\oint_{\gamma_i} dx^\mu \left(\partial_\mu\left(\partial^c\left[\frac{1}{|\vec{x}-\vec{y}|}\right]\right)\right)}_{\to 0}. \tag{4.14}$$

Therefore, the GLN can be alternatively written as follows

$$L(i,j) = \oint_{\gamma_i} dx^\mu \int_{S[\gamma_j,\vec{y}]} d\Sigma_\mu \delta^3(\vec{x}-\vec{y}). \tag{4.15}$$

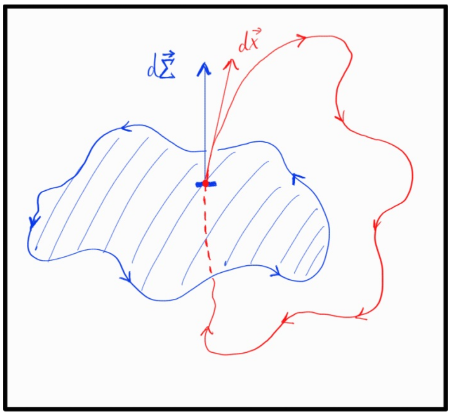

Figure 4.7: Gauss linking number (+)

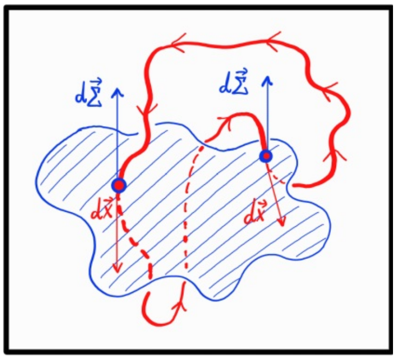

Figure 4.9: A pair of curves with GLN equal to two.

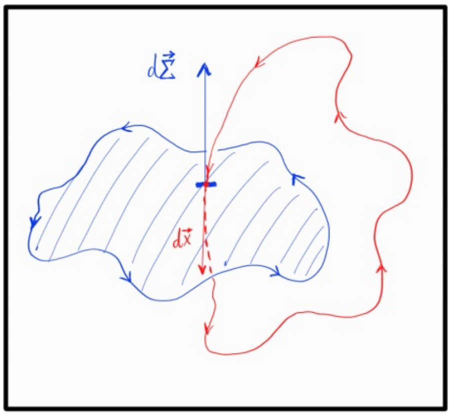

Figure 4.8: Gauss linking number (-).

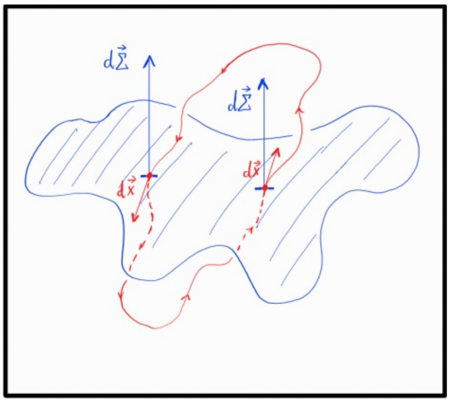

Figure 4.10: Trivial link (intersections cancel each other out).

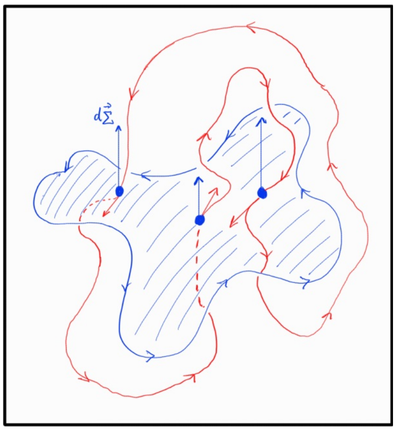

Figure 4.11: Deformation of a pair of curves with GLN equal to one.

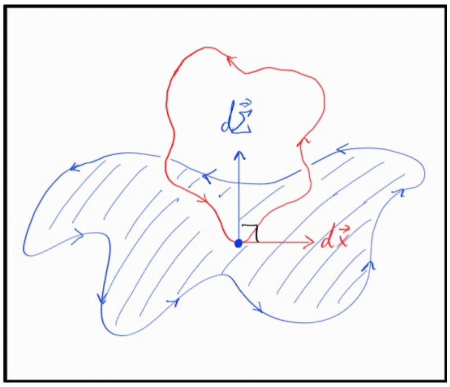

Figure 4.12: Trivial link (intersection at 90°).

Note that equation (4.15) tells us that curve γ_i (red) intersects with at least one point of the surface enclosed by curve γ_j (blue). We define situations where the two vectors have an angle between them less than 90° with a sign (+) as shown in Figure 4.7, and for a pair of vectors with an angle greater than 90° with a sign (-) as shown in Figure 4.8. In this way, it is possible to sum over all the oriented intersections between the curve γ_i and the surface enclosed by the curve γ_j, then take the absolute value of the result, and this number will correspond to the GLN between these curves. For example, two negative intersections will give us a GLN equal to two as shown in Figure 4.9. If the red curve intersects the blue surface more than one time, all the intersection must have the same orientation, otherwise we get the trivial link (if the intersections cancel each other out) as shown in Figure 4.10 or a deformation of a pair of curves with a GLN less than the number of intersections as we show in Figure 4.11. Additionally, the tangent vector to curve γ_i must form an angle different from 90° with respect to curve γ_j, otherwise the curves are not linked as depicted in Figure 4.12.

When comparing equations (2.74), (2.75), and (4.15), note that we can write

$$D_{j\ \mu x} = \frac{1}{4\pi}\oint_{\gamma_j} dz^{\rho}\varepsilon_{\mu\nu\rho}\frac{(x-z)^{\nu}}{|x-z|^3} \quad \leftrightarrow \quad \int_{S[\gamma_j,\vec{y}]} d\Sigma_{\mu}\delta^3(\vec{x}-\vec{y}), \tag{4.16}$$

which will allow us to carry out the geometric interpretation of the analytical expressions for the TMC and the four-components link.

4.2.2 Borromean rings

If we use (4.16) into the first term of the equation (2.61), which represents the analytical expression for $\bar{\mu}(1,2,3)$, we can write it in terms of surface integrals as follows

$$\begin{aligned} S_A^{(1)} &= \int d^3x \varepsilon^{\mu\nu\rho} D_{1\ \mu x} D_{2\ \nu x} D_{3\ \rho x} \\ &= \int d^3x \varepsilon^{\mu\nu\rho} \int_{S[\gamma_1,\vec{y}_1]} d\Sigma_\mu \int_{S[\gamma_2,\vec{y}_2]} d\Sigma_\nu \int_{S[\gamma_3,\vec{y}_3]} d\Sigma_\rho \delta^3\left(\vec{x}-\vec{y}_1\right) \delta^3\left(\vec{x}-\vec{y}_2\right) \delta^3\left(\vec{x}-\vec{y}_3\right). \end{aligned} \tag{4.17}$$

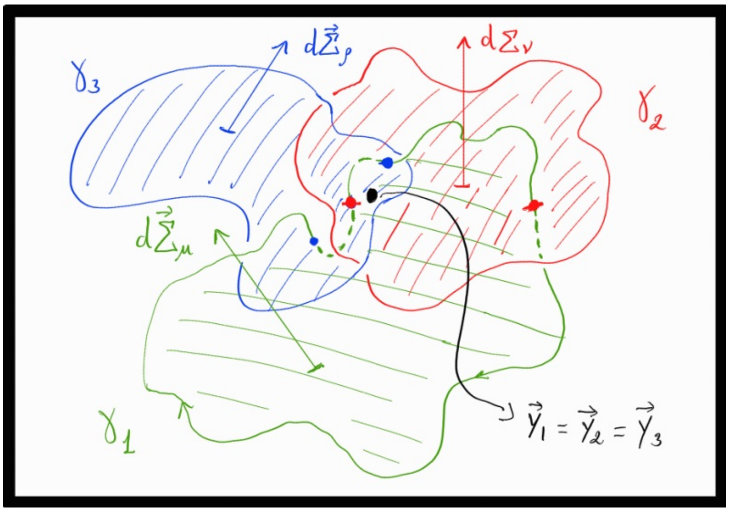

Figure 4.13: Borromean rings, contribution of the first term.

Equation (4.17) states that the three surfaces enclosed by the curves γ_1 (green), γ_2 (red), and γ_3 (blue) share a common point as shown in Figure 4.13. Furthermore, at this point, the vectors normal to these surfaces must have a nonzero mixed product, indicating that they are not coplanar vectors.

Similarly, the second, third, and fourth term of (2.61) take the form of the following expression,

$$\begin{aligned} S_B^{(1)} &= \int d^3x \int d^3y T_i^{\mu x\ \nu y} D_{j\ \mu x} D_{k\ \nu y} \\ &= \oint_{\gamma_i} dz_1^\mu \int_0^{z_1} dz_2^\nu \int_{S[\gamma_j,\vec{z}_3]} d\Sigma_\mu \int_{S[\gamma_k,\vec{z}_4]} d\Sigma_\nu \delta^3\left(\vec{z}_1-\vec{z}_3\right) \delta^3\left(\vec{z}_2-\vec{z}_4\right) \end{aligned} \tag{4.18}$$

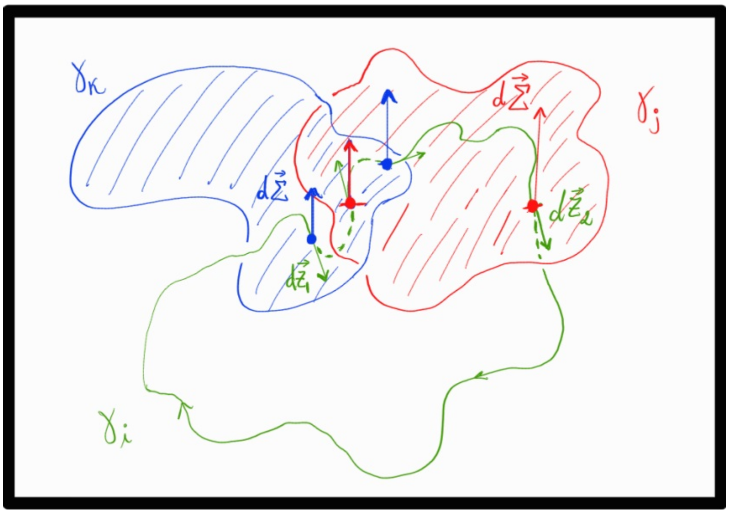

Figure 4.14: Borromean rings, contribution of the second, third, and fourth term.

Expression (4.18) states that curve γ_i (green) intersects at one point on the surface enclosed by curve γ_k (blue), and the surface enclosed by curve γ_j (red), in that order. Note that the outermost integral (in green) is a closed line integral, so curve γ_i must return to the point where it started. However, the innermost integral (in green) is an open line integral that advances ahead of the outermost integral. Therefore, before returning, curve γ_i must cut the surface enclosed by both curves γ_k and γ_j, both in the opposite direction, thus closing the line integral as depicted in Figure 4.14. If we view the surfaces enclosed by the curves γ_k (blue) and γ_j (red) as two fabrics and the curve γ_i (green) as a thread, then equation (4.18) is telling us how to use the thread to sew these fabrics, in other words, it shows us how the curves γ_i, γ_j, and γ_k are linked.

4.2.3 Four-components link

Note that $S^{(2)}$ (2.90) can be written in terms of the D-objects using equation (2.74), and integrating with respect to the contracted variables, as

$$
\begin{aligned}
S^{(2)} = \; & \frac{1}{8} \sum_{i,j,k,l} \left(\left(\vec{I}_i \times \left(\vec{I}_j \times \vec{I}_k \right) \right) \cdot \vec{I}_l \right) g_{\mu x \; \alpha y} \\
& \Bigg\{ \frac{3}{8} \int d^3x \int d^3y \varepsilon^{\mu\nu\rho} \varepsilon^{\alpha\beta\gamma} D_{i \; \nu x} D_{j \; \beta y} D_{k \; \gamma y} D_{l \; \rho x} \\
& - \frac{3}{2} \int d^3x \int d^3y \int d^3x_1 \varepsilon^{\mu\nu\rho} T_j^{[\alpha y \; \mu_1 x_1]} D_{i \; \nu x} D_{k \; \mu_1 x_1} D_{l \; \rho x} \\
& + \frac{3}{2} \int d^3x \int d^3y \int d^3x_1 \int d^3x_2 T_i^{[\mu x \; \mu_1 x_1]} T_j^{[\alpha y \; \mu_2 x_2]} D_{k \; \mu_2 x_2} D_{l \; \mu_1 x_1} \\
& + \int d^3x \int d^3y \int d^3x_1 \int d^3x_2 T_j^{\mu x \; \mu_1 x_1 \; \mu_2 x_2} D_{i \; \mu_1 x_1} D_{k \; \mu_2 x_2} T_l^{\alpha y} \Bigg\}.
\end{aligned}
\tag{4.19}
$$

Observe that in equation (4.19), there is a metric g that could not be contracted with a T-object of one-index to form a D-object. Such a metric can be replaced by $-g_{\mu x\ \nu y} \to H_{\mu\nu}\left(x, \gamma^{\vec{y}}\right)$, as in the open paths interpretation, but the difference lies in that for this new interpretation, the open paths have a much more intuitive geometric meaning than in the method of the previous section. Here, the open paths represent the crossing of one surface over another (like a kind of "twist"), as will be seen later when we analyze each contribution from (4.19). For this reason, this particular case is special because it is a combination of everything we have seen before.

The first term of (4.19) is given by

$$\begin{aligned} S_A^{(2)} &= \int d^3x \int d^3y \varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\mu x\ \alpha y} D_{i\ \nu x} D_{j\ \beta y} D_{k\ \gamma y} D_{l\ \rho x} \\ &= -\int_{\gamma^{\vec{x}_2}} dz^\rho \int_{S[\gamma_i,\vec{x}_1]} d\Sigma_\nu \int_{S[\gamma_j,\vec{x}_2]} d\Sigma_\beta \int_{S[\gamma_k,\vec{x}_3]} d\Sigma_\gamma \int_{S[\gamma_l,\vec{x}_4]} d\Sigma_\rho \varepsilon^{\nu\beta\gamma} \delta^3(\vec{z}-\vec{x}_1)\delta^3(\vec{x}_2-\vec{x}_3)\delta^3(\vec{z}-\vec{x}_4) \\ &+ \int_{\gamma^{\vec{x}_2}} dz^\nu \int_{S[\gamma_i,\vec{x}_1]} d\Sigma_\nu \int_{S[\gamma_j,\vec{x}_2]} d\Sigma_\beta \int_{S[\gamma_k,\vec{x}_3]} d\Sigma_\gamma \int_{S[\gamma_l,\vec{x}_4]} d\Sigma_\rho \varepsilon^{\rho\beta\gamma} \delta^3(\vec{z}-\vec{x}_1)\delta^3(\vec{x}_2-\vec{x}_3)\delta^3(\vec{z}-\vec{x}_4). \end{aligned} \tag{4.20}$$

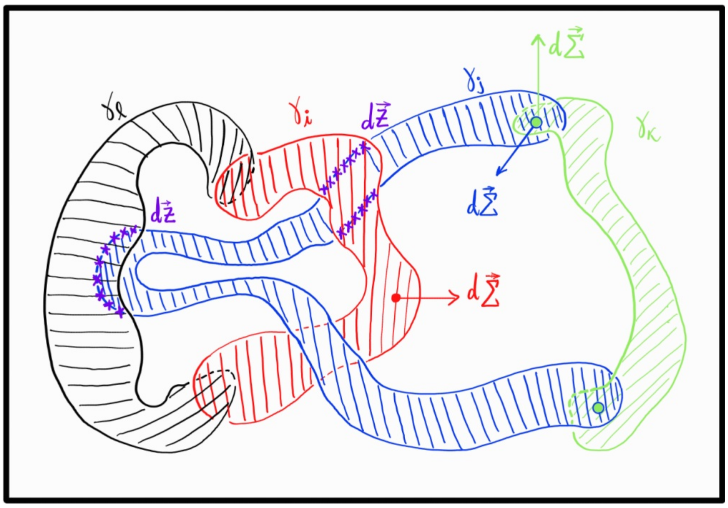

Figure 4.15: Four-components link, first term contribution.

Note that equation (4.20) states that the surfaces enclosed by the curves γ_j (blue) and γ_k (green) have points in common, meaning they intersect each other. Additionally, the contraction of the Levi-Civita symbol with the normal vectors to the surfaces enclosed by the γ_i (red), γ_j (blue), and γ_k (green) curves must form a non-degenerate volume, as shown in Figure 4.15. Finally, the line integral of open paths, which come from infinity and end at the edge of the blue curve γ_j (shown as violet $\times$ in Figure 4.15), represents crossings of the surfaces enclosed by the γ_i (red) and γ_l (black) curves over the γ_j (blue). In other words, the violet parallel paths (perpendicular to the screen) cut through the red and black surfaces because they must pass through them to end at the edge of the blue surface. If the red and black surfaces did not cross over the blue surface, then these open paths would never cut through them, and the curves would not be knotted. Note that because the indices of the curves γ_i, γ_j, γ_k, γ_l are just labels, and in equation (4.19) we sum over all indices, these conditions must be satisfied for each of the curves. In this way, it is possible to ensure that they are linked.

Now, the second term is given by

$$\begin{aligned} S_B^{(2)} &= \int d^3x \int d^3y \int d^3x_1 \varepsilon^{\mu\nu\rho} g_{\mu x\ \alpha y} T_j^{\alpha y\ \mu_1 x_1} D_{i\ \nu x} D_{k\ \mu_1 x_1} D_{l\ \rho x} \\ &= -\int_{\gamma^{\vec{z}_1}} dz^{\rho} \oint_{\gamma_j} dz_1^{\nu} \int_0^{z_1} dz_2^{\mu_1} \int_{S[\gamma_i,\vec{x}_2]} d\Sigma_{\nu} \int_{S[\gamma_k,\vec{x}_3]} d\Sigma_{\mu_1} \int_{S[\gamma_l,\vec{x}_4]} d\Sigma_{\rho} \delta^3(\vec{z}-\vec{x}_2)\delta^3(\vec{z}_2-\vec{x}_3)\delta^3(\vec{z}-\vec{x}_4) \\ &+ \int_{\gamma^{\vec{z}_1}} dz^{\nu} \oint_{\gamma_j} dz_1^{\rho} \int_0^{z_1} dz_2^{\mu_1} \int_{S[\gamma_i,\vec{x}_2]} d\Sigma_{\nu} \int_{S[\gamma_k,\vec{x}_3]} d\Sigma_{\mu_1} \int_{S[\gamma_l,\vec{x}_4]} d\Sigma_{\rho} \delta^3(\vec{z}-\vec{x}_2)\delta^3(\vec{z}_2-\vec{x}_3)\delta^3(\vec{z}-\vec{x}_4). \end{aligned} \tag{4.21}$$

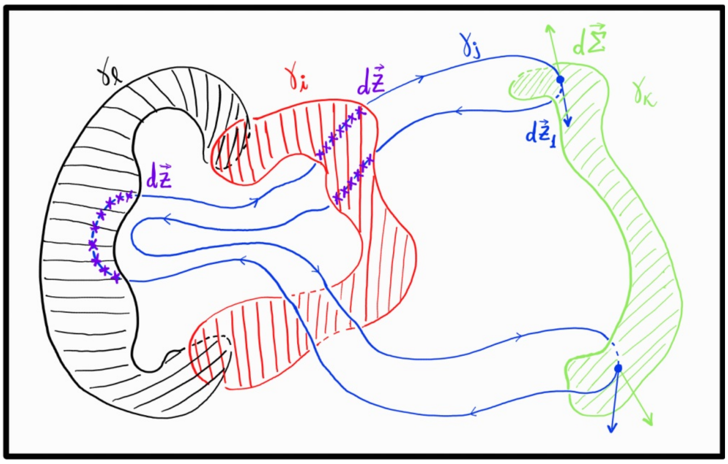

Figure 4.16: Four-components link, second term contribution.

Equation (4.21) states that the blue curve γ_j must intersect the surface enclosed by the curve γ_k (green) in one direction and then intersect it in the opposite direction before completely closing the path integral. Once again, terms with path integrals of open paths appear, representing crossings of the surfaces enclosed by the γ_i (red) and γ_l (black) curves over the blue one γ_j (similar to the previous case), as shown in Figure 4.16.

The third term is given by

$$\begin{aligned} S_C^{(2)} &= \int d^3x \int d^3y \int d^3x_1 \int d^3x_2 g_{\mu x\ \alpha y} T_i^{\mu x\ \mu_1 x_1} T_j^{\alpha y\ \mu_2 x_2} D_{k\ \mu_2 x_2} D_{l\ \mu_1 x_1} \\ &= \int_{\gamma^{\vec{z}_3}} dz^{\lambda} \oint_{\gamma_i} dz_1^{\mu} \int_0^{z_1} dz_2^{\mu_1} \oint_{\gamma_j} dz_3^{\alpha} \int_0^{z_3} dz_4^{\mu_2} \int_{S[\gamma_k, \vec{z}_5]} d\Sigma_{\mu_2} \int_{S[\gamma_l, \vec{z}_6]} d\Sigma_{\mu 1} \varepsilon_{\mu\alpha\lambda} \times \\ &\times \delta^3(\vec{z} - \vec{z}_1)\delta^3(\vec{z}_4 - \vec{z}_5)\delta^3(\vec{z}_2 - \vec{z}_6). \end{aligned} \tag{4.22}$$

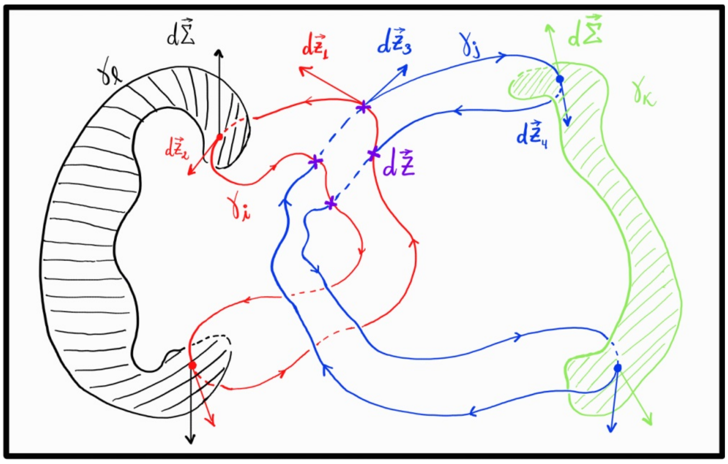

Figure 4.17: Four-components link, third term contribution.

Equation (4.22) states that the blue curve γ_j intersects the surface enclosed by the curve γ_k (green) in one direction and then intersects it in the opposite direction to close the path integral. Similarly, with the red curve γ_i, which it intersects the surface enclosed by the curve γ_l (black) in one direction and then re-intersects in the opposite direction to close the path integral. Note that we also have a line integral of open paths ending on the blue curve γ_j, which cuts through the red curve γ_i, representing the crossing of the red curve over the blue one (because otherwise, they are not linked). Finally, the tangent vectors to the curves γ_i (red), γ_j (blue), and the violet open paths must form a non-degenerate volume. Note that the tangent vectors to the open strands enter the screen and are represented by $\times$, as depicted in Figure 4.17.

Finally, the fourth term is given by

$$\begin{aligned} S_D^{(2)} &= \int d^3x \int d^3y \int d^3x_1 \int d^3x_2 T_j^{\mu x\ \mu_1 x_1\ \mu_2 x_2} D_{i\ \mu_1 x_1} D_{k\ \mu_2 x_2} T_l^{\alpha y} g_{\mu x\ \alpha y} \\ &= -\oint_{\gamma_j} dz_1^{\mu} \int_0^{z_1} dz_2^{\mu_1} \int_0^{z_2} dz_3^{\mu_2} \int_{S[\gamma_i, \vec{z}_4]} d\Sigma_{\mu_1} \int_{S[\gamma_k, \vec{z}_5]} d\Sigma_{\mu_2} \int_{S[\gamma_l, \vec{z}_6]} d\Sigma_{\mu} \times \\ &\times \delta^3(\vec{z}_1 - \vec{z}_6)\delta^3(\vec{z}_2 - \vec{z}_4)\delta^3(\vec{z}_3 - \vec{z}_5). \end{aligned} \tag{4.23}$$

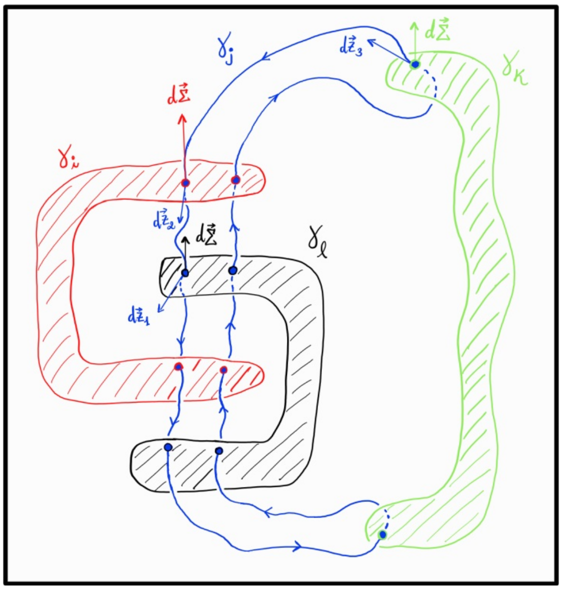

Figure 4.18: Four-components link, fourth term contribution.

Equation (4.23) states that the blue curve γ_j must intersect the surfaces enclosed by the curves γ_k (green), γ_i (red), and γ_l (black), as shown by the blue integrals, from the innermost integral to the outermost one. Now, since the outermost integral is a closed path integral, it must return to the initial point. To achieve this, it must intersect the green surface again in the opposite direction. However, as the two subsequent inner integrals (in blue) are open integrals that precede the closed path integral, the blue curve γ_j must first intersect in the opposite direction to the surfaces enclosed by the curves γ_l (black) and γ_i (red) before reaching the initial point, thus closing the path integral as shown in Figure 4.18. Note that this particular term can be seen as a thread crossing over three fabrics of different colors; in other words, the integral in (4.23) tells us how to weave the blue curve over the other three surfaces.

Conclusions

Throughout this work, we have delved into the study of topological classical field theories, focusing on the Chern-Simons-Wong theory. By solving the classical equations of motion, we obtained analytical expressions for various link invariants, including the Gauss linking number, the third Milnor's coefficient, and the four-components link.

One key aspect of our analysis involved perturbative methods to obtain these link invariants. At zeroth order, the Gauss linking number was identified, while at the first order, the third Milnor's coefficient was revealed, capturing the entanglement of Borromean rings. This perturbative approach, although non-trivial, provided insight into the intricate link structures associated with the Chern-Simons-Wong theory.

Additionally, we explored the possibility of intermediate Abelian theories that could reproduce higher-order link invariants, akin to those obtained from the non-Abelian Chern-Simons theory. Two sections were dedicated to this exploration: one based on the work of Leal and Pineda [3], which focused on an Abelian theory reproducing the TMC, and another introducing an original Abelian theory detecting the 4-component link.

Moreover, we developed a geometric interpretation for the link invariants using surface integrals. This approximation allowed us to have a more intuitive understanding than the interpretation provided by the open paths diagram, because the knot invariants were expressed through the relationships that the normal vectors to the involved surfaces satisfied among themselves. In conclusion, our work contributes to the broader understanding of topological classical field theories, showcasing the rich link invariants that arise from the Chern-Simons-Wong theory and its interpretations.

This work leaves the door open for future interesting research. For example, it is thought that the action $S^{(2)}$ could be related to the Whitehead link when restricted to two particles with independent iso-charges [9], which can be analyzed using the new interpretation of surface integrals. Additionally, a more intriguing question would be whether a Feynman-like diagrammatic approach could be developed to obtain analytical expressions for higher-order link invariants without having to calculate each perturbative term of the on-shell action $S^{(p)}$.

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Appendix A

Intermediate action for the four-components link

A.1 Intermediate action and equations of motion

The intermediate action for the four-components link in contracted notation is given by:

$$
\begin{aligned}
S_M = & -6\vec{\Lambda}_{\mu x}\left[\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\lambda}_{\rho x} - \frac{1}{2}\sum_i^n T_i^{\mu x}\vec{I}_i(0)\right] \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\mu x\ \alpha y}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\cdot\vec{\lambda}_{\rho x} \\
& + \frac{3}{2}\sum_{i=1}^n\varepsilon^{\mu\nu\rho}g_{\mu x\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\vec{\lambda}_{\rho x} \\
& + \frac{3}{4}\sum_{i,j}g_{\mu x\ \alpha y}T_i^{[\mu_1 x_1\ \mu x]}T_j^{[\alpha y\ \mu_2 x_2]}\left[\vec{I}_i\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\right]\cdot\vec{\lambda}_{\mu_1 x_1} \\
& + \sum_{i=1}^n T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\vec{\lambda}_{\mu x},
\end{aligned}
\tag{A.1}
$$

where we have used the convention of generalized Einstein summation for "discrete-continuous" indices. Transitioning from vector notation to index notation for the fields Λ and λ, we get:

$$\begin{aligned} S_M = & -6\Lambda^a_{\mu x}\left[\varepsilon^{\mu\nu\rho}\partial_\nu\lambda^a_{\rho x} - \frac{1}{2}\sum_i^n T_i^{\mu x} I_i^a(0)\right] \\ & + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\ \alpha y}\lambda^b_{\nu x}\lambda^d_{\beta y}\lambda^e_{\gamma y}\lambda^a_{\rho x} \\ & + \frac{3}{2}\sum_{i=1}^n \varepsilon^{\mu\nu\rho}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\lambda^b_{\nu x}\lambda^d_{\mu_1 x_1} I_i^e \lambda^a_{\rho x} \\ & + \frac{3}{4}\sum_{i,j}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\ \alpha y} T_i^{[\mu_1 x_1\ \mu x]} T_j^{[\alpha y\ \mu_2 x_2]} I_i^b \lambda^d_{\mu_2 x_2} I_j^e \lambda^a_{\mu_1 x_1} \\ & + \sum_{i=1}^n \varepsilon^{abc}\varepsilon^{cde} T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\lambda^b_{\mu_1 x_1}\lambda^d_{\mu_2 x_2} I_i^e \lambda^a_{\mu x}. \end{aligned} \tag{A.2}$$

If we make variations of the action ($\delta S_M = 0$) with respect to the field $\Lambda^a_{\mu x}$, we trivially obtain:

$$\varepsilon^{\mu\nu\rho}\partial_\nu\lambda^a_{\rho x} = \frac{1}{2}\sum_i^n T_i^{\mu x} I_i^a(0), \tag{A.3}$$

which is the zeroth-order equation of motion for the Chern-Simons-Wong theory, as in [5]. When performing variations of the action with respect to the field $\lambda^a_{\mu x}$, note that:

$$\begin{aligned}
\delta S_M = & -6\varepsilon^{\mu\nu\rho}\Lambda^a_{\mu x}\partial_\nu\left(\delta\lambda^a_{\rho x}\right) \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}\left(\delta\lambda^b_{\nu x}\right)\lambda^d_{\beta y}\lambda^e_{\gamma y}\lambda^a_{\rho x} \leftarrow \ (b\leftrightarrow a)^\wedge(\nu\leftrightarrow\rho) \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}\lambda^b_{\nu x}\left(\delta\lambda^d_{\beta y}\right)\lambda^e_{\gamma y}\lambda^a_{\rho x} \leftarrow (d\leftrightarrow a)^\wedge(x\leftrightarrow y)^\wedge(\mu\nu\rho)\leftrightarrow(\alpha\beta\gamma) \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}\lambda^b_{\nu x}\lambda^d_{\beta y}\left(\delta\lambda^e_{\gamma y}\right)\lambda^a_{\rho x} \leftarrow (e\leftrightarrow a)^\wedge(x\leftrightarrow y)^\wedge(\mu\nu\rho)\leftrightarrow(\alpha\beta\gamma) \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}\lambda^b_{\nu x}\lambda^d_{\beta y}\lambda^e_{\gamma y}\left(\delta\lambda^a_{\rho x}\right) \\
& + \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\left(\delta\lambda^b_{\nu x}\right)\lambda^d_{\mu_1 x_1}I^e_i\lambda^a_{\rho x} \leftarrow (b\leftrightarrow a)^\wedge(\nu\leftrightarrow\rho) \\
& + \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\lambda^b_{\nu x}\left(\delta\lambda^d_{\mu_1 x_1}\right)I^e_i\lambda^a_{\rho x} \leftarrow (d\leftrightarrow a)^\wedge(\mu_1 x_1\leftrightarrow\rho x) \\
& + \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\lambda^b_{\nu x}\lambda^d_{\mu_1 x_1}I^e_i\left(\delta\lambda^a_{\rho x}\right) \\
& + \frac{3}{4}\sum_{i,j}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}T_i^{[\mu_1 x_1\ \mu x]}T_j^{[\alpha y\ \mu_2 x_2]}I^b_i\left(\delta\lambda^d_{\mu_2 x_2}\right)I^e_j\lambda^a_{\mu_1 x_1} \leftarrow (d\leftrightarrow a)^\wedge(\mu_2 x_2\leftrightarrow\mu_1 x_1) \\
& + \frac{3}{4}\sum_{i,j}\varepsilon^{abc}\varepsilon^{cde}g_{\mu x\ \alpha y}T_i^{[\mu_1 x_1\ \mu x]}T_j^{[\alpha y\ \mu_2 x_2]}I^b_i\lambda^d_{\mu_2 x_2}I^e_j\left(\delta\lambda^a_{\mu_1 x_1}\right) \\
& + \sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde}T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left(\delta\lambda^b_{\mu_1 x_1}\right)\lambda^d_{\mu_2 x_2}I^e_i\lambda^a_{\mu x} \leftarrow (b\leftrightarrow a)^\wedge\left(\mu_1 x_1\leftrightarrow\mu x\right) \\
& + \sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde}T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\lambda^b_{\mu_1 x_1}\left(\delta\lambda^d_{\mu_2 x_2}\right)I^e_i\lambda^a_{\mu x} \leftarrow (d\leftrightarrow a)^\wedge\left(\mu_2 x_2\leftrightarrow\mu x\right) \\
& + \sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde}T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\lambda^b_{\mu_1 x_1}\lambda^d_{\mu_2 x_2}I^e_i\left(\delta\lambda^a_{\mu x}\right).
\end{aligned} \tag{A.4}$$

Also, note that

$$\text{(boundary term)} \rightarrow \ \cancelto{0}{\partial_\nu\left(\Lambda^a_{\mu x}\delta\lambda^a_{\rho x}\right)} = \Lambda^a_{\mu x}\partial_\nu\left(\delta\lambda^a_{\rho x}\right) + \left(\partial_\nu\Lambda^a_{\mu x}\right)\delta\lambda^a_{\rho x},$$

$$\Rightarrow \Lambda^a_{\mu x}\partial_\nu\left(\delta\lambda^a_{\rho x}\right) = -\left(\partial_\nu\Lambda^a_{\mu x}\right)\delta\lambda^a_{\rho x}, \tag{A.5}$$

so we can write

$$
\begin{aligned}
\delta S_M =\ & 6\varepsilon^{\mu\nu\rho}\left(\partial_\nu \Lambda^a_{\mu x}\right)\delta\lambda^a_{\rho x} \leftarrow (\mu \leftrightarrow \rho) \\
& + \frac{3}{4}\varepsilon^{\mu\rho\nu}\varepsilon^{\alpha\beta\gamma}\varepsilon^{bac}\varepsilon^{cde} g_{\mu x\ \alpha y}\left(\delta\lambda^a_{\rho x}\right)\lambda^d_{\beta y}\lambda^e_{\gamma y}\lambda^b_{\nu x} \leftarrow \left(\varepsilon^{\mu\rho\nu} = -\varepsilon^{\mu\nu\rho}\right)^\wedge \left(\varepsilon^{bac} = -\varepsilon^{abc}\right) \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{dbc}\varepsilon^{cae} g_{\alpha y\ \mu x}\lambda^b_{\beta y}\left(\delta\lambda^a_{\nu x}\right)\lambda^e_{\rho x}\lambda^d_{\gamma y} \leftarrow (\nu \leftrightarrow \rho)^\wedge (b \leftrightarrow d)^\wedge (e \leftrightarrow b) \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{ebc}\varepsilon^{cda} g_{\alpha y\ \mu x}\lambda^b_{\beta y}\lambda^d_{\nu x}\left(\delta\lambda^a_{\rho x}\right)\lambda^e_{\gamma y} \leftarrow (b \leftrightarrow d) \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\ \alpha y}\lambda^b_{\nu x}\lambda^d_{\beta y}\lambda^e_{\gamma y}\left(\delta\lambda^a_{\rho x}\right) \\
& + \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\mu\rho\nu}\varepsilon^{bac}\varepsilon^{cde} g_{\mu x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\left(\delta\lambda^a_{\rho x}\right)\lambda^d_{\mu_1 x_1} I^e_i \lambda^b_{\nu x} \leftarrow \left(\varepsilon^{\mu\rho\nu} = -\varepsilon^{\mu\nu\rho}\right)^\wedge \left(\varepsilon^{bac} = -\varepsilon^{abc}\right)^\wedge (\mu \leftrightarrow \rho) \\
& + \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\mu\nu\mu_1}\varepsilon^{dbc}\varepsilon^{cae} g_{\mu x_1\ \alpha y} T_i^{[\alpha y\ \rho x]}\lambda^b_{\nu x_1}\left(\delta\lambda^a_{\rho x}\right) I^e_i \lambda^d_{\mu_1 x_1} \leftarrow (\mu \leftrightarrow \rho) \\
& + \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\lambda^b_{\nu x}\lambda^d_{\mu_1 x_1} I^e_i \left(\delta\lambda^a_{\rho x}\right) \leftarrow (\mu \leftrightarrow \rho) \\
& + \frac{3}{4}\sum_{i,j}\varepsilon^{dbc}\varepsilon^{cae} g_{\mu x\ \alpha y} T_i^{[\mu_2 x_2\ \mu x]} T_j^{[\alpha y\ \mu_1 x_1]} I^b_i \left(\delta\lambda^a_{\mu_1 x_1}\right) I^e_j \lambda^d_{\mu_2 x_2} \leftarrow (\mu_1 x_1 \leftrightarrow \mu x) \\
& + \frac{3}{4}\sum_{i,j}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\ \alpha y} T_i^{[\mu_1 x_1\ \mu x]} T_j^{[\alpha y\ \mu_2 x_2]} I^b_i \lambda^d_{\mu_2 x_2} I^e_j \left(\delta\lambda^a_{\mu_1 x_1}\right) \leftarrow (\mu_1 x_1 \leftrightarrow \mu x) \\
& + \sum_{i=1}^{n}\varepsilon^{bac}\varepsilon^{cde} T_i^{\mu_1 x_1\ \mu x\ \mu_2 x_2}\left(\delta\lambda^a_{\mu x}\right)\lambda^d_{\mu_2 x_2} I^e_i \lambda^b_{\mu_1 x_1} \leftarrow \left(\varepsilon^{bac} = -\varepsilon^{abc}\right) \\
& + \sum_{i=1}^{n}\varepsilon^{dbc}\varepsilon^{cae} T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\lambda^b_{\mu_1 x_1}\left(\delta\lambda^a_{\mu x}\right) I^e_i \lambda^d_{\mu_2 x_2} \\
& + \sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde} T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\lambda^b_{\mu_1 x_1}\lambda^d_{\mu_2 x_2} I^e_i \left(\delta\lambda^a_{\mu x}\right).
\end{aligned} \tag{A.6}
$$

Upon making the respective changes in each line (note that we can combine the terms from lines six and eight), so we obtain

$$
\begin{aligned}
\delta S_M = \; & 6\varepsilon^{\rho\nu\mu}\left(\partial_\nu \Lambda^a_{\rho x}\right)\delta\lambda^a_{\mu x} \leftarrow \left(\varepsilon^{\rho\nu\mu} = -\varepsilon^{\mu\nu\rho}\right) \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\alpha y}\left(\delta\lambda^a_{\rho x}\right)\lambda^d_{\beta y}\lambda^e_{\gamma y}\lambda^b_{\nu x} \\
& + \frac{3}{4}\varepsilon^{\mu\rho\nu}\varepsilon^{\alpha\beta\gamma}\varepsilon^{edc}\varepsilon^{cab} g_{\alpha y}\mu x\lambda^d_{\beta y}\left(\delta\lambda^a_{\rho x}\right)\lambda^e_{\gamma y}\lambda^b_{\nu x} \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{edc}\varepsilon^{cba} g_{\alpha y\mu x}\lambda^d_{\beta y}\lambda^e_{\gamma y}\left(\delta\lambda^a_{\rho x}\right)\lambda^b_{\nu x} \\
& + \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\alpha y}\lambda^b_{\nu x}\lambda^d_{\beta y}\lambda^e_{\gamma y}\left(\delta\lambda^a_{\rho x}\right) \\
& + 3\sum_{i=1}^{n}\varepsilon^{\rho\nu\mu}\varepsilon^{abc}\varepsilon^{cde} g_{\rho x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\lambda^b_{\nu x}\lambda^d_{\mu_1 x_1} I^e_i\left(\delta\lambda^a_{\mu x}\right) \\
& + \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\rho\nu\mu_1}\varepsilon^{dbc}\varepsilon^{cae} g_{\rho x_1\ \alpha y} T_i^{[\alpha y\ \mu x]}\lambda^b_{\nu x_1}\left(\delta\lambda^a_{\mu x}\right) I^e_i\lambda^d_{\mu_1 x_1} \leftarrow \left(\alpha y \leftrightarrow \mu_1 z_1\right)^\wedge \left(x_1 \leftrightarrow y\right)^\wedge \left(\rho\nu\mu_1 \leftrightarrow \alpha\beta\gamma\right) \\
& + \frac{3}{4}\sum_{i,j}\varepsilon^{dbc}\varepsilon^{cae} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu_2 x_2\ \mu_1 x_1]} T_j^{[\alpha y\ \mu x]} I^b_i\left(\delta\lambda^a_{\mu x}\right) I^e_j\lambda^d_{\mu_2 x_2} \leftarrow \left(i \leftrightarrow j\right)^\wedge \left(\alpha y \leftrightarrow \mu_1 x_1\right)^\wedge \left(b \leftrightarrow e\right) \\
& + \frac{3}{4}\sum_{i,j}\varepsilon^{abc}\varepsilon^{cde} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]} I^b_i\lambda^d_{\mu_2 x_2} I^e_j\left(\delta\lambda^a_{\mu x}\right) \\
& - \sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde} T_i^{\mu_1 x_1\ \mu x\ \mu_2 x_2}\left(\delta\lambda^a_{\mu x}\right)\lambda^d_{\mu_2 x_2} I^e_i\lambda^b_{\mu_1 x_1} \\
& + \sum_{i=1}^{n}\varepsilon^{dbc}\varepsilon^{cae} T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\lambda^b_{\mu_1 x_1}\left(\delta\lambda^a_{\mu x}\right) I^e_i\lambda^d_{\mu_2 x_2} \\
& + \sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde} T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\lambda^b_{\mu_1 x_1}\lambda^d_{\mu_2 x_2} I^e_i\left(\delta\lambda^a_{\mu x}\right).
\end{aligned}
\tag{A.7}
$$

If we define the following object

$$
T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2} = \frac{1}{2}\left(T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2} - T_i^{\mu_1 x_1\ \mu x\ \mu_2 x_2}\right), \tag{A.8}
$$

note that we can add up the terms from lines two to five of equation (A.7), and using the definition in (A.8), we have

$$
\begin{aligned}
\delta S_M =& -6\varepsilon^{\mu\nu\rho}\left(\partial_\nu \Lambda^a_{\rho x}\right)\delta\lambda^a_{\mu x} \\
&+3\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde} g_{\mu x\ \alpha y}\lambda^b_{\nu x}\lambda^d_{\beta y}\lambda^e_{\gamma y}\left(\delta\lambda^a_{\rho x}\right)\leftarrow(\mu\leftrightarrow\rho) \\
&+3\sum_{i=1}^{n}\varepsilon^{\rho\nu\mu}\varepsilon^{abc}\varepsilon^{cde} g_{\rho x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\lambda^b_{\nu x}\lambda^d_{\mu_1 x_1} I^e_i\left(\delta\lambda^a_{\mu x}\right)\leftarrow\left(\varepsilon^{\rho\nu\mu}=-\varepsilon^{\mu\nu\rho}\right) \\
&+\frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\alpha\beta\gamma}\varepsilon^{cbd}\varepsilon^{ace} g_{\alpha y\ \mu_1 z_1} T_i^{[\mu_1 z_1\ \mu x]}\lambda^b_{\beta y}\lambda^d_{\gamma y} I^e_i\left(\delta\lambda^a_{\mu x}\right)\leftarrow\left(T_i^{[\mu_1 z_1\ \mu x]}=-T_i^{[\mu x\ \mu_1 z_1]}\right)^{\wedge}(z_1\leftrightarrow x_1) \\
&+\frac{3}{4}\sum_{i,j}\varepsilon^{dec}\varepsilon^{cab} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu_1 x_1\ \mu x]_1} T_j^{[\mu_2 x_2\ \alpha y]} I^e_j I^b_i\lambda^d_{\mu_2 x_2}\left(\delta\lambda^a_{\mu x}\right)\leftarrow\left(T_i^{[\mu_1 x_1\ \mu x]}=-T_i^{[\mu x\ \mu_1 x_1]}\right) \\
&+\frac{3}{4}\sum_{i,j}\varepsilon^{abc}\varepsilon^{cde} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]} I^b_i\lambda^d_{\mu_2 x_2} I^e_j\left(\delta\lambda^a_{\mu x}\right) \\
&+2\sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde} T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2}\lambda^b_{\mu_1 x_1}\lambda^d_{\mu_2 x_2} I^e_i\left(\delta\lambda^a_{\mu x}\right) \\
&+\sum_{i=1}^{n}\varepsilon^{dbc}\varepsilon^{cae} T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\lambda^b_{\mu_1 x_1} I^e_i\lambda^d_{\mu_2 x_2}\left(\delta\lambda^a_{\mu x}\right),
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
\delta S_M =& -6\varepsilon^{\mu\nu\rho}\left(\partial_\nu \Lambda^a_{\rho x}\right)\delta\lambda^a_{\mu x} \\
&+3\varepsilon^{\rho\nu\mu}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde} g_{\rho x\ \alpha y}\lambda^b_{\nu x}\lambda^d_{\beta y}\lambda^e_{\gamma y}\left(\delta\lambda^a_{\mu x}\right)\leftarrow\left(\varepsilon^{\rho\nu\mu}=-\varepsilon^{\mu\nu\rho}\right) \\
&-3\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho}\varepsilon^{abc}\varepsilon^{cde} g_{\rho x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\lambda^b_{\nu x}\lambda^d_{\mu_1 x_1} I^e_i\left(\delta\lambda^a_{\mu x}\right) \\
&-\frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\alpha\beta\gamma}\varepsilon^{cbd}\varepsilon^{ace} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]}\lambda^b_{\beta y}\lambda^d_{\gamma y} I^e_i\left(\delta\lambda^a_{\mu x}\right) \\
&+\frac{3}{2}\sum_{i,j}\varepsilon^{abc}\varepsilon^{cde} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]} I^b_i\lambda^d_{\mu_2 x_2} I^e_j\left(\delta\lambda^a_{\mu x}\right)\leftarrow\left(\varepsilon^{abc}=-\varepsilon^{acb}\right) \\
&+2\sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde} T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2}\lambda^b_{\mu_1 x_1}\lambda^d_{\mu_2 x_2} I^e_i\left(\delta\lambda^a_{\mu x}\right) \\
&+\sum_{i=1}^{n}\varepsilon^{dbc}\varepsilon^{cae} T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\lambda^b_{\mu_1 x_1} I^e_i\lambda^d_{\mu_2 x_2}\left(\delta\lambda^a_{\mu x}\right),
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
\delta S_M =& - 6\varepsilon^{\mu\nu\rho}\left(\partial_\nu \Lambda^a_{\rho x}\right)\delta\lambda^a_{\mu x} \\
&- 3\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\varepsilon^{abc}\varepsilon^{cde} g_{\rho x\ \alpha y}\lambda^b_{\nu x}\lambda^d_{\beta y}\lambda^e_{\gamma y}\left(\delta\lambda^a_{\mu x}\right) \\
&- 3\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho}\varepsilon^{abc}\varepsilon^{cde} g_{\rho x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\lambda^b_{\nu x}\lambda^d_{\mu_1 x_1} I^e_i\left(\delta\lambda^a_{\mu x}\right) \\
&- \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\alpha\beta\gamma}\varepsilon^{cbd}\varepsilon^{ace} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]}\lambda^b_{\beta y}\lambda^d_{\gamma y} I^e_i\left(\delta\lambda^a_{\mu x}\right) \\
&- \frac{3}{2}\sum_{i,j}\varepsilon^{acb}\varepsilon^{cde} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]} I^b_i \lambda^d_{\mu_2 x_2} I^e_j\left(\delta\lambda^a_{\mu x}\right) \\
&+ 2\sum_{i=1}^{n}\varepsilon^{abc}\varepsilon^{cde} T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2}\lambda^b_{\mu_1 x_1}\lambda^d_{\mu_2 x_2} I^e_i\left(\delta\lambda^a_{\mu x}\right) \\
&+ \sum_{i=1}^{n}\varepsilon^{dbc}\varepsilon^{cae} T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\lambda^b_{\mu_1 x_1} I^e_i \lambda^d_{\mu_2 x_2}\left(\delta\lambda^a_{\mu x}\right).
\end{aligned}
\tag{A.11}
$$

The last equation can be expressed using the vector notation as

$$
\begin{aligned}
\delta S_M =& - 6\varepsilon^{\mu\nu\rho}\left(\partial_\nu \vec{\Lambda}_{\rho x}\right)\cdot\left(\delta\vec{\lambda}_{\mu x}\right) \\
&- 3\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\rho x\ \alpha y}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right) \\
&- 3\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho} g_{\rho x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right) \\
&- \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\alpha\beta\gamma} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]}\left[\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\times\vec{I}_i\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right) \\
&- \frac{3}{2}\sum_{i,j} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]}\left[\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\times\vec{I}_i\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right) \\
&+ 2\sum_{i=1}^{n} T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right) \\
&+ \sum_{i=1}^{n} T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{\lambda}_{\mu_2 x_2}\right)\times\vec{I}_i\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right),
\end{aligned}
\tag{A.12}
$$

and observe that the last term could be rewritten as follows

$$
T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{\lambda}_{\mu_2 x_2}\right)\times\vec{I}_i\right] = -T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\vec{I}_i\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{\lambda}_{\mu_2 x_2}\right)\right], \tag{A.13}
$$

using the Jacobi identity, we have

$$
\begin{aligned}
T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{\lambda}_{\mu_2 x_2}\right)\times\vec{I}_i\right] =& \ T^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\vec{\lambda}_{\mu_2 x_2}\times\left(\vec{I}_i\times\vec{\lambda}_{\mu_1 x_1}\right)\right] \leftarrow (\mu_2 x_2 \leftrightarrow \mu_1 x_1) \\
&+ T^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right],
\end{aligned}
\tag{A.14}
$$

$$T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{\lambda}_{\mu_2 x_2}\right)\times\vec{I}_i\right]= T^{\mu_1 x_1\ \mu_2 x_2\ \mu x}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{I}_i\times\vec{\lambda}_{\mu_2 x_2}\right)\right] + T^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right], \tag{A.15}$$

$$T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{\lambda}_{\mu_2 x_2}\right)\times\vec{I}_i\right]=- T^{\mu_1 x_1\ \mu_2 x_2\ \mu x}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right] + T^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right], \tag{A.16}$$

and using the definition in (A.8), we have

$$T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\left[\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{\lambda}_{\mu_2 x_2}\right)\times\vec{I}_i\right]=-2T^{[\mu_1 x_1\ \mu_2 x_2]\ \mu x}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]. \tag{A.17}$$

Replacing equation (A.17) into (A.12),

$$\begin{aligned}
\delta S_M =& -6\varepsilon^{\mu\nu\rho}\left(\partial_\nu\vec{\Lambda}_{\rho x}\right)\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&-3\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\rho x\ \alpha y}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&-3\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho}g_{\rho x\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&-\frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\alpha\beta\gamma}g_{\mu_1 x_1\ \alpha y}T_i^{[\mu x\ \mu_1 x_1]}\left[\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\times\vec{I}_i\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&-\frac{3}{2}\sum_{i,j}g_{\mu_1 x_1\ \alpha y}T_i^{[\mu x\ \mu_1 x_1]}T_j^{[\alpha y\ \mu_2 x_2]}\left[\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\times\vec{I}_i\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&+2\sum_{i=1}^{n}T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&-2\sum_{i=1}^{n}T_i^{[\mu_1 x_1\ \mu_2 x_2]\ \mu x}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right).
\end{aligned} \tag{A.18}$$

If we define the object

$$T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}=\frac{2}{3}\left(T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2}-T_i^{[\mu_1 x_1\ \mu_2 x_2]\ \mu x}\right), \tag{A.19}$$

so we can write the variation of the action as

$$
\begin{aligned}
\delta S_M = &- 6\varepsilon^{\mu\nu\rho}\left(\partial_\nu \vec{\Lambda}_{\rho x}\right)\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&- 3\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\rho x\ \alpha y}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&- 3\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho} g_{\rho x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&- \frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\alpha\beta\gamma} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]}\left[\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\times\vec{I}_i\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&- \frac{3}{2}\sum_{i,j} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]}\left[\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\times\vec{I}_i\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right)\\
&+ 3\sum_{i=1}^{n} T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\left(\delta\vec{\lambda}_{\mu x}\right),
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
\delta S_M = &- 3\Bigg\{2\varepsilon^{\mu\nu\rho}\partial_\nu \vec{\Lambda}_{\rho x}\\
&+ \varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\rho x\ \alpha y}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\\
&+ \sum_{i=1}^{n}\varepsilon^{\mu\nu\rho} g_{\rho x\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\\
&+ \frac{1}{2}\sum_{i=1}^{n}\varepsilon^{\alpha\beta\gamma} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]}\left[\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\times\vec{I}_i\right]\\
&+ \frac{1}{2}\sum_{i,j} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]}\left[\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\times\vec{I}_i\right]\\
&- \sum_{i=1}^{n} T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\Bigg\}\cdot\left(\delta\vec{\lambda}_{\mu x}\right).
\end{aligned}
\tag{A.21}
$$

Finally, by the least action principle ($\delta S_M = 0$), we get the equation of motion for the Λ field:

$$
\begin{aligned}
2\varepsilon^{\mu\nu\rho}\partial_\nu \vec{\Lambda}_{\rho x} = &- \varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} g_{\rho\bar{x}\ \alpha y}\left[\vec{\lambda}_{\nu\bar{x}}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\\
&- \sum_{i=1}^{n}\varepsilon^{\mu\nu\rho} g_{\rho\bar{x}\ \alpha y} T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu\bar{x}}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\\
&- \frac{1}{2}\sum_{i=1}^{n}\varepsilon^{\alpha\beta\gamma} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]}\left[\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\times\vec{I}_i\right]\\
&- \frac{1}{2}\sum_{i,j} g_{\mu_1 x_1\ \alpha y} T_i^{[\mu x\ \mu_1 x_1]} T_j^{[\alpha y\ \mu_2 x_2]}\left[\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\times\vec{I}_i\right]\\
&+ \sum_{i=1}^{n} T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right],
\end{aligned}
\tag{A.22}
$$

which is analogous to the second-order equation of motion for the Chern-Simons-Wong theory [5]. This is evident when grouping terms one-two and three-four in the above equation. Thus, we obtain:

$$\begin{aligned}2\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\Lambda}_{\rho x} =& -2\varepsilon^{\mu\nu\rho}\vec{\lambda}_{\nu\bar{x}}\times\left(\frac{1}{2}\varepsilon^{\alpha\beta\gamma}\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)g_{\rho\bar{x}\ \alpha y}+\frac{1}{2}\sum_{i=1}^{n}T_i^{[\alpha y\ \mu_1 x_1]}\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)g_{\rho\bar{x}\ \alpha y}\right)\\ &-\sum_{i=1}^{n}T_i^{[\mu x\mu_1 x_1]}\left(\frac{1}{2}\varepsilon^{\alpha\beta\gamma}\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)g_{\mu_1 x_1\ \alpha y}+\frac{1}{2}\sum_{j=1}^{n}T_j^{[\alpha y\ \mu_2 x_2]}\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)g_{\mu_1 x_1\ \alpha y}\right)\times\vec{I}_i\\ &+\sum_{i=1}^{n}T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right],\end{aligned} \tag{A.23}$$

If we define the object

$$\vec{a}_{\mu x}=\frac{1}{2}\varepsilon^{\alpha\beta\gamma}\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)g_{\mu x\ \alpha y}+\frac{1}{2}\sum_{i=1}^{n}T_i^{[\alpha y\ \mu_1 x_1]}\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)g_{\mu x\ \alpha y}, \tag{A.24}$$

then equation (A.22) can be written as

$$\begin{aligned}2\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\Lambda}_{\rho x} =& -2\varepsilon^{\mu\nu\rho}\vec{\lambda}_{\nu\bar{x}}\times\vec{a}_{\rho\bar{x}}-\sum_{i=1}^{n}T_i^{[\mu x\ \mu_1 x_1]}\vec{a}_{\mu_1 x_1}\times\vec{I}_i\\ &+\sum_{i=1}^{n}T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right],\end{aligned} \tag{A.25}$$

and changing the indices ($\nu\leftrightarrow\rho$) in the first term on the right-hand side of the above equation, and considering ($\epsilon^{\mu\rho\nu}=-\epsilon^{\mu\nu\rho}$), we have

$$\begin{aligned}2\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\Lambda}_{\rho x} =& -2\varepsilon^{\mu\nu\rho}\vec{a}_{\nu\bar{x}}\times\vec{\lambda}_{\rho\bar{x}}-\sum_{i=1}^{n}T_i^{[\mu x\ \mu_1 x_1]}\vec{a}_{\mu_1 x_1}\times\vec{I}_i\\ &+\sum_{i=1}^{n}T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right],\end{aligned} \tag{A.26}$$

which is formally equal to the second-order equation of motion shown in [5]. Note that the object $\vec{a}_{\mu x}$ is an analogue to the first-order field in the Chern-Simons-Wong theory, which has naturally appeared in this intermediate Abelian theory. In fact, by multiplying equation (A.24) by the inverse of the metric $g^{\mu x\ \nu y}$ and using $g^{\mu x\ \nu y}\vec{a}_{\nu y}=-\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}_{\rho x}$ with $g^{\mu x\ \alpha z}g_{\alpha z\ \nu y}=\delta^{\mu x}_{\nu y}$, note that:

$$\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}_{\rho x}=-\frac{1}{2}\varepsilon^{\mu\nu\rho}\left(\vec{\lambda}_{\nu\bar{x}}\times\vec{\lambda}_{\rho\bar{x}}\right)-\frac{1}{2}\sum_{i=1}^{n}T_i^{[\mu x\ \mu_1 x_1]}\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right). \tag{A.27}$$

If we remove the vector notation, perform the variable change ($\mu_1 x_1\leftrightarrow\nu y$) in the second term on the right-hand side, and revert the Einstein summation convention, we obtain

$$\varepsilon^{\mu\nu\rho}\partial_\nu a_{i\rho}(x) = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{ijk}\lambda_{j\nu}(x)\lambda_{k\rho}(x) - \frac{1}{2}\int d^3y \sum_{l=1}^{n} \varepsilon^{ijk} T_l^{[\mu x\ \nu y]}\lambda_{j\nu}(y) I_{lk}, \tag{A.28}$$

and considering a set of orthonormal iso-currents $I_{lk} = \delta_{lk}$, we have

$$\varepsilon^{\mu\nu\rho}\partial_\nu a_{i\rho}(x) = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{ijk}\lambda_{j\nu}(x)\lambda_{k\rho}(x) - \frac{1}{2}\int d^3y \varepsilon^{ijk} T_k^{[\mu x\ \nu y]}\lambda_{j\nu}(y), \tag{A.29}$$

changing the indices $(j \leftrightarrow k)$ in the second term on the right-hand side of the previous equation and considering that $(\epsilon_{ikj} = -\epsilon^{ijk})$, we obtain:

$$\varepsilon^{\mu\nu\rho}\partial_\nu a_{i\rho}(x) = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{ijk}\lambda_{j\nu}(x)\lambda_{k\rho}(x) + \frac{1}{2}\int d^3y \varepsilon^{ijk} T_j^{[\mu x\ \nu y]}\lambda_{k\nu}(y), \tag{A.30}$$

which is the equation of motion for the A field of the intermediate theory proposed in [3] and, in turn, analogous to the first-order equation of motion of the Chern-Simons-Wong theory [5].

A.2 On-shell action:

By replacing (A.3) and (A.22) into (A.1):

$$\begin{aligned} S_{M\ On-Shell} =\ & \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\mu x\ \alpha y}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\beta y}\times\vec{\lambda}_{\gamma y}\right)\right]\cdot\vec{\lambda}_{\rho x} \\ & +\frac{3}{2}\sum_{i=1}^{n}\varepsilon^{\mu\nu\rho}g_{\mu x\ \alpha y}T_i^{[\alpha y\ \mu_1 x_1]}\left[\vec{\lambda}_{\nu x}\times\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right)\right]\cdot\vec{\lambda}_{\rho x}\leftarrow(i\leftrightarrow j) \\ & +\frac{3}{4}\sum_{i,j}g_{\mu x\ \alpha y}T_i^{[\mu_1 x_1\ \mu x]}T_j^{[\alpha y\ \mu_2 x_2]}\left[\vec{I}_i\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_j\right)\right]\cdot\vec{\lambda}_{\mu_1 x_1} \\ & +\sum_{i=1}^{n}T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1}\times\left(\vec{\lambda}_{\mu_2 x_2}\times\vec{I}_i\right)\right]\cdot\vec{\lambda}_{\mu x}\leftarrow(i\leftrightarrow j), \end{aligned} \tag{A.31}$$

$$\begin{aligned} S_{M\ On-Shell} =\ & \frac{3}{4}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\mu x\ \alpha y}\frac{1}{16}\sum_{i,j,k,l}\left(\vec{I}_i\times\left(\vec{I}_j\times\vec{I}_k\right)\right)\cdot\vec{I}_l g_{\nu x\ \mu_1 x_1}g_{\beta y\ \mu_2 x_2}g_{\gamma y\ \mu_3 x_3}g_{\rho x\ \mu_4 x_4}T_i^{\mu_1 x_1}T_j^{\mu_2 x_2}T_\kappa^{\mu_3 x_3}T_l^{\mu_4 x_4} \\ & -\frac{3}{2}\sum_{j=1}^{n}\varepsilon^{\mu\nu\rho}g_{\mu x\ \alpha y}T_j^{[\alpha y\ \mu_1 x_1]}\frac{1}{8}\sum_{i,k,l}\left(\vec{I}_i\times\left(\vec{I}_j\times\vec{I}_k\right)\right)\cdot\vec{I}_l g_{\nu x\ \mu_2 x_2}g_{\mu_1 x_1\ \mu_3 x_3}g_{\rho x\ \mu_4 x_4}T_i^{\mu_2 x_2}T_k^{\mu_3 x_3}T_l^{\mu_4 x_4} \\ & -\frac{3}{4}\sum_{i,j}g_{\mu x\ \alpha y}T_i^{[\mu_1 x_1\ \mu x]}T_j^{[\alpha y\ \mu_2 x_2]}\frac{1}{4}\sum_{k,l}\left(\vec{I}_i\times\left(\vec{I}_j\times\vec{I}_k\right)\right)\cdot\vec{I}_l g_{\mu_2 x_2\ \mu_3 x_3}g_{\mu_1 x_1\ \mu_4 x_4}T_k^{\mu_3 x_3}T_l^{\mu_4 x_4} \\ & -\sum_{j=1}^{n}T_j^{\mu x\ \mu_1 x_1\ \mu_2 x_2}\frac{1}{8}\sum_{i,k,l}\left(\vec{I}_i\times\left(\vec{I}_j\times\vec{I}_k\right)\right)\cdot\vec{I}_l g_{\mu_1 x_1\ \mu_3 x_3}g_{\mu_2 x_2\ \mu_4 x_4}g_{\mu x\ \alpha y}T_i^{\mu_3 x_3}T_k^{\mu_4 x_4}T_l^{\alpha y} \end{aligned} \tag{A.32}$$

$$\begin{aligned} S_{M\ On-Shell} =\ & \frac{1}{8}\sum_{i,j,k,l}\left(\left(\vec{I}_i\times\left(\vec{I}_j\times\vec{I}_k\right)\right)\cdot\vec{I}_l\right)\left\{\frac{3}{8}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}g_{\nu x\ \mu_1 x_1}g_{\beta y\ \mu_2 x_2}g_{\gamma y\ \mu_3 x_3}g_{\rho x\ \mu_4 x_4}T_i^{\mu_1 x_1}T_j^{\mu_2 x_2}T_\kappa^{\mu_3 x_3}T_l^{\mu_4 x_4}\right. \\ & +\frac{3}{2}\varepsilon^{\mu\nu\rho}T_j^{[\alpha y\ \mu_1 x_1]}g_{\nu x\ \mu_2 x_2}g_{\mu_1 x_1\ \mu_3 x_3}g_{\rho x\ \mu_4 x_4}T_i^{\mu_2 x_2}T_k^{\mu_3 x_3}T_l^{\mu_4 x_4} \\ & +\frac{3}{2}T_i^{[\mu x\ \mu_1 x_1]}T_j^{[\alpha y\ \mu_2 x_2]}g_{\mu_2 x_2\ \mu_3 x_3}g_{\mu_1 x_1\ \mu_4 x_4}T_k^{\mu_3 x_3}T_l^{\mu_4 x_4} \\ & \left.+T_j^{\mu x\ \mu_1 x_1\ \mu_2 x_2}g_{\mu_1 x_1\ \mu_3 x_3}g_{\mu_2 x_2\ \mu_4 x_4}T_i^{\mu_3 x_3}T_k^{\mu_4 x_4}T_l^{\alpha y}\right\}g_{\mu x\ \alpha y}, \end{aligned} \tag{A.33}$$

which is exactly the same as the second-order "on-shell" term of the action for the Chern-Simons-Wong model [5].

A.3 Consistency of the theory:

Taking the divergence of equation (A.3), note that

$$\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\lambda}_{\rho x} = J^{\mu x} = \frac{1}{2}\sum_{i=1}^{n} T_i^{\mu x}\vec{I}_i, \tag{A.34}$$

$$\text{(divergence of the curl)} \rightarrow \ \cancelto{0}{\partial_\mu\left(\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\lambda}_{\rho x}\right)} = \partial_\mu J^{\mu x} = \frac{1}{2}\sum_{i}^{m}\left(\partial_\mu T_i^{\mu x}\right)\vec{I}_i, \tag{A.35}$$

Recall that in general the function of the currents is not zero, we have

$$\partial_\mu T_i^{\mu x} = 0 \ \ \rightarrow \text{consistent by the differential constraint.} \tag{A.36}$$

Taking the divergence of equation (A.27):

$$\cancelto{0}{\partial_\mu\left(\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}_{\rho x}\right)} = -\frac{1}{2}\underbrace{\left(\varepsilon^{\mu\nu\rho}\partial_\mu\vec{\lambda}_{\nu\bar{x}}\right)\times\vec{\lambda}_{\rho\bar{x}}}_{(\mu\leftrightarrow\nu)\wedge(\rho\leftrightarrow\mu)} - \frac{1}{2}\underbrace{\vec{\lambda}_{\nu\bar{x}}\times\left(\varepsilon^{\mu\nu\rho}\partial_\mu\vec{\lambda}_{\rho\bar{x}}\right)}_{(\mu\leftrightarrow\nu)} - \frac{1}{2}\sum_{i=1}^{n}\partial_\mu T_i^{[\mu x\ \mu_1 x_1]}\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right). \tag{A.37}$$

$$0 = -\frac{1}{2}\left(\varepsilon^{\nu\rho\mu}\partial_\nu\vec{\lambda}_{\rho\bar{x}}\right)\times\vec{\lambda}_{\mu\bar{x}} - \frac{1}{2}\vec{\lambda}_{\mu\bar{x}}\times\left(\varepsilon^{\nu\mu\rho}\partial_\nu\vec{\lambda}_{\rho\bar{x}}\right) - \frac{1}{2}\sum_{i=1}^{n}\partial_\mu T_i^{[\mu x\ \mu_1 x_1]}\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right). \tag{A.38}$$

Combining similar terms

$$0 = \vec{\lambda}_{\mu\bar{x}}\times\left(\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\lambda}_{\rho\bar{x}}\right) - \frac{1}{2}\sum_{i=1}^{n}\partial_\mu T_i^{[\mu x\ \mu_1 x_1]}\left(\vec{\lambda}_{\mu_1 x_1}\times\vec{I}_i\right). \tag{A.39}$$

Observe that

$$\begin{aligned}
\partial_\mu T_i^{[\mu x\ \mu_1 x_1]} &= \frac{1}{2}\partial_\mu T_i^{\mu x\ \mu_1 x_1} - \frac{1}{2}\partial_\mu T_i^{\mu_1 x_1\ \mu x} \\
&= \frac{1}{2}\left(-\delta^3\left(x-x_i\right)+\delta^3\left(x-x_1\right)\right)T_i^{\mu_1 x_1} - \frac{1}{2}\left(-\delta^3\left(x-x_1\right)+\delta^3\left(x-x_i\right)\right)T_i^{\mu_1 x_1} \\
&= \left(-\delta^3\left(x-x_i\right)+\delta^3\left(x-x_1\right)\right)T_i^{\mu_1 x_1}.
\end{aligned} \tag{A.40}$$

Replacing the previous result and the equation of motion for the λ field, we have

$$0=\frac{1}{2}\sum_{i=1}^{n}T_i^{\mu\bar{x}}\left(\vec{\lambda}_{\mu\bar{x}}\times\vec{I}_i\right)+\frac{1}{2}\sum_{i=1}^{n}\delta^3\left(x-x_i\right)T_i^{\mu_1x_1}\left(\vec{\lambda}_{\mu_1x_1}\times\vec{I}_i\right)-\frac{1}{2}\sum_{i=1}^{n}\delta^3\left(x-x_1\right)T_i^{\mu_1x_1}\left(\vec{\lambda}_{\mu_1x_1}\times\vec{I}_i\right), \tag{A.41}$$

integrate with respect to x_1 in the third term, and changing the indices $(\mu_1\leftrightarrow\mu)$, $(i\leftrightarrow j)$:

$$0=\cancel{\frac{1}{2}\sum_{i=1}^{n}T_i^{\mu\bar{x}}\left(\vec{\lambda}_{\mu\bar{x}}\times\vec{I}_i\right)}+\frac{1}{2}\sum_{i=1}^{n}\delta^3\left(x-x_i\right)T_i^{\mu_1x_1}\left(\vec{\lambda}_{\mu_1x_1}\times\vec{I}_i\right)-\cancel{\frac{1}{2}\sum_{i=1}^{n}T_i^{\mu\bar{x}}\left(\vec{\lambda}_{\mu\bar{x}}\times\vec{I}_i\right)}, \tag{A.42}$$

$$\Rightarrow\quad \frac{1}{2}\sum_{j=1}^{n}\delta^3\left(x-x_j\right)T_j^{\mu_1x_1}\left(\vec{\lambda}_{\mu_1x_1}\times\vec{I}_j\right)=0. \tag{A.43}$$

Recall that $\lambda_{\mu x}=-g_{\mu x\;\nu y}J^{\nu y}$, and replacing it with the expression for current density

$$\Rightarrow\quad \frac{1}{2}\sum_{j=1}^{n}\delta^3\left(x-x_j\right)T_j^{\mu_1x_1}\left(\left(\frac{1}{2}\sum_{i=1}^{n}g_{\mu_1x_1\;\mu_2x_2}T_i^{\mu_2x_2}\vec{I}_i\right)\times\vec{I}_j\right)=0, \tag{A.44}$$

$$\frac{1}{4}\sum_{i,j}\delta^3\left(x-x_j\right)(I_i\times I_j)T_j^{\mu_1x_1}g_{\mu_1x_1\;\mu_2x_2}T_i^{\mu_2x_2}=0\quad\leftarrow(\mu_1x_1\leftrightarrow\mu_2x_2), \tag{A.45}$$

$$\frac{1}{4}\sum_{i,j}\delta^3\left(x-x_j\right)(I_i\times I_j)T_i^{\mu_1x_1}g_{\mu_1x_1\;\mu_2x_2}T_j^{\mu_2x_2}=0, \tag{A.46}$$

$$\frac{1}{4}\sum_{i,j}\delta^3\left(x-x_j\right)(I_i\times I_j)L(i,j)=0. \tag{A.47}$$

In general, the function of the currents does not always vanish, so it holds that:

$$L(i,j)=0, \tag{A.48}$$

which would be our first important result for the consistency of this theory, since if the proposed action describes a higher-order link invariant, then the Gauss linking number must be zero.

Now, we take the divergence of equation (A.25):

$$\begin{aligned}\cancelto{0}{2\partial_\mu\left(\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\Lambda}_{\rho x}\right)}=&\cancelto{A}{-2\varepsilon^{\mu\nu\rho}\left(\partial_\mu\vec{\lambda}_{\nu\bar{x}}\right)\times\vec{a}_{\rho\bar{x}}}\cancelto{B}{-2\varepsilon^{\mu\nu\rho}\vec{\lambda}_{\nu\bar{x}}\times\left(\partial_\mu\vec{a}_{\rho\bar{x}}\right)}\\&-\sum_{i=1}^{n}\cancelto{C}{\left(\partial_\mu T_i^{[\mu x\;\mu_1x_1]}\right)}\left(\vec{a}_{\mu_1x_1}\times\vec{I}_i\right)\\&+\sum_{i=1}^{n}\cancelto{D}{\left(\partial_\mu T_i^{\langle[\mu x\;\mu_1x_1]\;\mu_2x_2\rangle}\right)}\left[\vec{\lambda}_{\mu_1x_1}\times\left(\vec{\lambda}_{\mu_2x_2}\times\vec{I}_i\right)\right].\end{aligned} \tag{A.49}$$

Note that

$$
\begin{aligned}
A = -2\left(\varepsilon^{\mu\nu\rho}\partial_\mu\vec{\lambda}_{\nu\bar{x}}\right)\times\vec{a}_{\rho\bar{x}} &= -2\left(\varepsilon^{\nu\mu\rho}\partial_\nu\vec{\lambda}_{\mu\bar{x}}\right)\times\vec{a}_{\rho\bar{x}} \quad \leftarrow(\mu\leftrightarrow\nu)\\
&= -2\left(\varepsilon^{\nu\rho\mu}\partial_\nu\vec{\lambda}_{\rho\bar{x}}\right)\times\vec{a}_{\mu\bar{x}} \quad \leftarrow(\mu\leftrightarrow\rho)\\
&= -2\left(\varepsilon^{\mu\nu\rho}\partial_\nu\vec{\lambda}_{\rho\bar{x}}\right)\times\vec{a}_{\mu\bar{x}}\\
&= -2\left(\frac{1}{2}\sum_{i=1}^{n}T_i^{\mu\bar{x}}\vec{I}_i\right)\times\vec{a}_{\mu\bar{x}}\\
&= -\sum_{i=1}^{n}T_i^{\mu\bar{x}}\left(\vec{I}_i\times\vec{a}_{\mu\bar{x}}\right),
\end{aligned}
\tag{A.50}
$$

$$
\begin{aligned}
B &= -2\varepsilon^{\mu\nu\rho}\vec{\lambda}_{\nu\bar{x}}\times\left(\partial_\mu\vec{a}_{\rho\bar{x}}\right) = -2\vec{\lambda}_{\nu\bar{x}}\times\left(\varepsilon^{\mu\nu\rho}\partial_\mu\vec{a}_{\rho\bar{x}}\right) \quad \leftarrow(\mu\leftrightarrow\nu)\\
&= -2\vec{\lambda}_{\mu\bar{x}}\times\left(\varepsilon^{\nu\mu\rho}\partial_\nu\vec{a}_{\rho\bar{x}}\right) = 2\vec{\lambda}_{\mu\bar{x}}\times\left(\varepsilon^{\mu\nu\rho}\partial_\nu\vec{a}_{\rho\bar{x}}\right) \quad \leftarrow \text{Replacing (A.27)}\\
&= -\underbrace{\varepsilon^{\mu\nu\rho}\vec{\lambda}_{\mu\bar{x}}\times\left(\vec{\lambda}_{\nu\bar{x}}\times\vec{\lambda}_{\rho\bar{x}}\right)}_{(\mu\leftrightarrow\rho)} - \sum_{i=1}^{n}\underbrace{T_i^{[\mu\bar{x}\ \mu_1x_1]}\vec{\lambda}_{\mu\bar{x}}\times\left(\vec{\lambda}_{\mu_1x_1}\times\vec{I}_i\right)}_{(\mu_1x_1\leftrightarrow\nu y)}\\
&= -\varepsilon^{\rho\nu\mu}\vec{\lambda}_{\rho\bar{x}}\times\left(\vec{\lambda}_{\nu\bar{x}}\times\vec{\lambda}_{\mu\bar{x}}\right) - \sum_{i=1}^{n}T_i^{[\mu\bar{x}\ \nu y]}\vec{\lambda}_{\mu\bar{x}}\times\left(\vec{\lambda}_{\nu y}\times\vec{I}_i\right)\\
&= \varepsilon^{\mu\nu\rho}\left(\vec{\lambda}_{\mu\bar{x}}\times\vec{\lambda}_{\nu\bar{x}}\right)\times\vec{\lambda}_{\rho\bar{x}} + \sum_{i=1}^{n}T_i^{[\mu\bar{x}\ \nu y]}\left(\vec{\lambda}_{\nu y}\times\vec{I}_i\right)\times\vec{\lambda}_{\mu\bar{x}},
\end{aligned}
\tag{A.51}
$$

$$
\begin{aligned}
C &= \partial_\mu T_i^{[\mu x\ \mu_1x_1]} = \frac{1}{2}\partial_\mu T_i^{\mu x\ \mu_1x_1} - \frac{1}{2}\partial_\mu T_i^{\mu_1x_1\ \mu x}\\
&= \frac{1}{2}\left(-\delta^3\left(x-x_i\right)+\delta^3\left(x-x_1\right)\right)T_i^{\mu_1x_1} - \frac{1}{2}\left(-\delta^3\left(x-x_1\right)+\delta^3\left(x-x_i\right)\right)T_i^{\mu_1x_1}\\
&= \left(-\delta^3\left(x-x_i\right)+\delta^3\left(x-x_1\right)\right)T_i^{\mu_1x_1},
\end{aligned}
\tag{A.52}
$$

$$
\begin{aligned}
D &= \partial_\mu T_i^{\langle[\mu x\ \mu_1 x_1]\ \mu_2 x_2\rangle} = \frac{2}{3}\left(\partial_\mu T_i^{[\mu x\ \mu_1 x_1]\ \mu_2 x_2} - \partial_\mu T_i^{[\mu_1 x_1\ \mu_2 x_2]\ \mu x}\right) \\
&= \frac{2}{3}\left(\frac{1}{2}\partial_\mu T_i^{\mu x\ \mu_1 x_1\ \mu_2 x_2} - \frac{1}{2}\partial_\mu T_i^{\mu_1 x_1\ \mu x\ \mu_2 x_2} - \frac{1}{2}\partial_\mu T_i^{\mu_1 x_1\ \mu_2 x_2\ \mu x} + \frac{1}{2}\partial_\mu T_i^{\mu_2 x_2\ \mu_1 x_1\ \mu x}\right) \\
&= \frac{1}{3}\Bigg(\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_1 x_1\ \mu_2 x_2} - \left(-\delta^3\left(x - x_1\right) + \cancel{\delta^3\left(x - x_2\right)}\right) T_i^{\mu_1 x_1\ \mu_2 x_2} \\
&- \left(-\cancel{\delta^3\left(x - x_2\right)} + \delta^3\left(x - x_i\right)\right) T_i^{\mu_1 x_1\ \mu_2 x_2} + \left(-\delta^3\left(x - x_1\right) + \delta^3\left(x - x_i\right)\right) T_i^{\mu_2 x_2\ \mu_1 x_1}\Bigg) \\
&= \frac{1}{3}\Bigg(\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_1 x_1\ \mu_2 x_2} - \left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_2 x_2\ \mu_1 x_1} \\
&+ \left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_1 x_1\ \mu_2 x_2}\Bigg) \\
&= \frac{1}{3}\Bigg(2\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{[\mu_1 x_1\ \mu_2 x_2]} + \underbrace{\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_1 x_1\ \mu_2 x_2}}_{T_i^{\mu_1 x_1\ \mu_2 x_2} = T_i^{[\mu_1 x_1\ \mu_2 x_2]} + T_i^{(\mu_1 x_1\ \mu_2 x_2)}}\Bigg) \\
&= \frac{1}{3}\Bigg(3\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{[\mu_1 x_1\ \mu_2 x_2]} + \underbrace{\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{(\mu_1 x_1\ \mu_2 x_2)}}_{T_i^{(\mu_1 x_1\ \mu_2 x_2)} = T_i^{\mu_1 x_1} T_i^{\mu_2 x_2}}\Bigg) \\
&= \left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{[\mu_1 x_1\ \mu_2 x_2]} + \frac{1}{3}\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_1 x_1} T_i^{\mu_2 x_2}.
\end{aligned}
\tag{A.53}
$$

Replace (A.50), (A.51), (A.52), and (A.53) into (A.49), we have

$$
\begin{aligned}
0 = &- \sum_{i=1}^{n} T_i^{\mu\bar{x}}\left(\vec{I}_i \times \vec{a}_{\mu\bar{x}}\right) + \varepsilon^{\mu\nu\rho}\left(\vec{\lambda}_{\mu\bar{x}} \times \vec{\lambda}_{\nu\bar{x}}\right) \times \vec{\lambda}_{\rho\bar{x}} + \sum_{i=1}^{n} T_i^{[\mu\bar{x}\ \nu y]}\left(\vec{\lambda}_{\nu y} \times \vec{I}_i\right) \times \vec{\lambda}_{\mu\bar{x}} \\
&- \sum_{i=1}^{n}\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_1 x_1}\left(\vec{a}_{\mu_1 x_1} \times \vec{I}_i\right) \\
&+ \sum_{i=1}^{n}\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T^{[\mu_1 x_1\ \mu_2 x_2]}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right] \\
&+ \frac{1}{3}\sum_{i=1}^{n}\left(-\delta^3\left(x - x_i\right) + \delta^3\left(x - x_1\right)\right) T_i^{\mu_1 x_1} T_i^{\mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right],
\end{aligned}
\tag{A.54}
$$

and integrate with respect to x_1 the terms containing $\delta^3(x-x_1)$, so we have

$$
\begin{aligned}
0 =& -\sum_{i=1}^{n} T_i^{\mu\bar{x}}\left(\vec{I}_i \times \vec{a}_{\mu\bar{x}}\right) + \varepsilon^{\mu\nu\rho}\left(\vec{\lambda}_{\mu\bar{x}} \times \vec{\lambda}_{\nu\bar{x}}\right) \times \vec{\lambda}_{\rho\bar{x}} + \sum_{i=1}^{n} T_i^{[\mu\bar{x}\ \nu y]}\left(\vec{\lambda}_{\nu y} \times \vec{I}_i\right) \times \vec{\lambda}_{\mu\bar{x}} \\
&+ \sum_{i=1}^{n} \delta^3(x-x_i)\, T_i^{\mu_1 x_1}\left(\vec{a}_{\mu_1 x_1} \times \vec{I}_i\right) - \sum_{i=1}^{n} \underbrace{T_i^{\mu_1\bar{x}}\left(\vec{a}_{\mu_1\bar{x}} \times \vec{I}_i\right)}_{(\mu_1\leftrightarrow\mu)} \\
&- \sum_{i=1}^{n} \delta^3(x-x_i)\, T^{[\mu_1 x_1\ \mu_2 x_2]}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right] + \sum_{i=1}^{n} \underbrace{T^{[\mu_1\bar{x}\ \mu_2 x_2]}\left[\vec{\lambda}_{\mu_1\bar{x}} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right]}_{(\mu_1\leftrightarrow\mu)^\wedge(\mu_2 x_2\longleftrightarrow\nu y)} \\
&- \frac{1}{3}\sum_{i=1}^{n} \delta^3(x-x_i)\, T_i^{\mu_1 x_1} T_i^{\mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right] + \frac{1}{3}\sum_{i=1}^{n} \underbrace{T_i^{\mu_1\bar{x}} T_i^{\mu_2 x_2}\left[\vec{\lambda}_{\mu_1\bar{x}} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right]}_{(\mu_1\leftrightarrow\mu)^\wedge(\mu_2 x_2\longleftrightarrow\nu y)},
\end{aligned}
\tag{A.55}
$$

$$
\begin{aligned}
0 =& -\cancel{\sum_{i=1}^{n} T_i^{\mu\bar{x}}\left(\vec{I}_i \times \vec{a}_{\mu\bar{x}}\right)} + \varepsilon^{\mu\nu\rho}\left(\vec{\lambda}_{\mu\bar{x}} \times \vec{\lambda}_{\nu\bar{x}}\right) \times \vec{\lambda}_{\rho\bar{x}} + \cancel{\sum_{i=1}^{n} T_i^{[\mu\bar{x}\ \nu y]}\left(\vec{\lambda}_{\nu y} \times \vec{I}_i\right)} \times \vec{\lambda}_{\mu\bar{x}} \\
&+ \sum_{i=1}^{n} \delta^3(x-x_i)\, T_i^{\mu_1 x_1}\left(\vec{a}_{\mu_1 x_1} \times \vec{I}_i\right) + \cancel{\sum_{i=1}^{n} T_i^{\mu\bar{x}}\left(\vec{I}_i \times \vec{a}_{\mu\bar{x}}\right)} \\
&- \sum_{i=1}^{n} \delta^3(x-x_i)\, T^{[\mu_1 x_1\ \mu_2 x_2]}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right] - \cancel{\sum_{i=1}^{n} T_i^{[\mu\bar{x}\ \nu y]}\left[\left(\vec{\lambda}_{\nu y} \times \vec{I}_i\right) \times \vec{\lambda}_{\mu\bar{x}}\right]} \\
&- \frac{1}{3}\sum_{i=1}^{n} \delta^3(x-x_i)\, T_i^{\mu_1 x_1} T_i^{\mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right] + \frac{1}{3}\sum_{i=1}^{n} T_i^{\mu\bar{x}} T_i^{\nu y}\left[\vec{\lambda}_{\mu\bar{x}} \times \left(\vec{\lambda}_{\nu y} \times \vec{I}_i\right)\right].
\end{aligned}
\tag{A.56}
$$

If we Simplify and organize terms, note that:

$$
\begin{aligned}
0 =& \varepsilon^{\mu\nu\rho}\left(\vec{\lambda}_{\mu\bar{x}} \times \vec{\lambda}_{\nu\bar{x}}\right) \times \vec{\lambda}_{\rho\bar{x}} + \sum_{i=1}^{n} \delta^3(x-x_i)\left\{T_i^{\mu_1 x_1}\left(\vec{a}_{\mu_1 x_1} \times \vec{I}_i\right) - T^{[\mu_1 x_1\ \mu_2 x_2]}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right]\right\} \\
&- \frac{1}{3}\sum_{i=1}^{n} \delta^3(x-x_i)\, T_i^{\mu_1 x_1} T_i^{\mu_2 x_2}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right] - \frac{1}{3}\sum_{i=1}^{n} T_i^{\mu\bar{x}} T_i^{\nu y}\left[\left(\vec{\lambda}_{\nu y} \times \vec{I}_i\right) \times \vec{\lambda}_{\mu\bar{x}}\right].
\end{aligned}
\tag{A.57}
$$

Due to the total anti-symmetry of $\varepsilon^{\mu\nu\rho}$, note that:

$$
\varepsilon^{\mu\nu\rho}\left(\vec{\lambda}_{\mu\bar{x}} \times \vec{\lambda}_{\nu\bar{x}}\right) \times \vec{\lambda}_{\rho\bar{x}} = \varepsilon^{\mu\nu\rho}\vec{\lambda}_{\nu\bar{x}} \underbrace{\left(\vec{\lambda}_{\mu\bar{x}} \cdot \vec{\lambda}_{\rho\bar{x}}\right)}_{\text{symmetric in } (\mu\rho)} - \varepsilon^{\mu\nu\rho}\vec{\lambda}_{\mu\bar{x}} \underbrace{\left(\vec{\lambda}_{\nu\bar{x}} \cdot \vec{\lambda}_{\rho\bar{x}}\right)}_{\text{symmetric in } (\nu\rho)} = 0,
\tag{A.58}
$$

so we obtain

$$0 = \sum_{i=1}^{n} \delta^3 (x - x_i) \left\{ T_i^{\mu_1 x_1} \left(\vec{a}_{\mu_1 x_1} \times \vec{I}_i \right) - T^{[\mu_1 x_1 \ \mu_2 x_2]} \left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i \right) \right] \right\}.$$
$$- \frac{1}{3} \sum_{i=1}^{n} \delta^3 (x - x_i) T_i^{\mu_1 x_1} T_i^{\mu_2 x_2} \left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i \right) \right] - \frac{1}{3} \sum_{i=1}^{n} T_i^{\mu \bar{x}} T_i^{\nu y} \left[\left(\vec{\lambda}_{\nu y} \times \vec{I}_i \right) \times \vec{\lambda}_{\mu \bar{x}} \right]. \tag{A.59}$$

Observe that

$$\begin{aligned} T_i^{\mu \bar{x}} T_i^{\nu y} \left[\vec{\lambda}_{\mu \bar{x}} \times \left(\vec{\lambda}_{\nu y} \times \vec{I}_i \right) \right] &= T_i^{\mu \bar{x}} T_i^{\nu y} \left[\left(\frac{1}{2} \sum_{k=1}^{n} g_{\mu \bar{x} \ \mu_1 x_1} T_k^{\mu_1 x_1} \vec{I}_k \right) \times \left(\left(\frac{1}{2} \sum_{j=1}^{n} g_{\nu y \ \rho z} T_j^{\rho z} \vec{I}_j \right) \times \vec{I}_i \right) \right] \\ &= \frac{1}{4} \sum_{j,k} (\vec{I}_k \times (\vec{I}_j \times \vec{I}_i)) g_{\mu \bar{x} \ \mu_1 x_1} g_{\nu y \ \rho z} T_i^{\mu \bar{x}} T_i^{\nu y} T_j^{\rho z} T_k^{\mu_1 x_1} \\ &= \frac{1}{4} \sum_{j,k} (\vec{I}_k \times (\vec{I}_j \times \vec{I}_i)) g_{\mu \bar{x} \ \mu_1 x_1} T_i^{\mu \bar{x}} (T_i^{\nu y} g_{\nu y \ \rho z} T_j^{\rho z}) T_k^{\mu_1 x_1} \\ &= \frac{1}{4} \sum_{j,k} (\vec{I}_k \times (\vec{I}_j \times \vec{I}_i)) g_{\mu \bar{x} \ \mu_1 x_1} T_i^{\mu \bar{x}} L(i,j) T_k^{\mu_1 x_1} = 0. \end{aligned} \tag{A.60}$$

Similarly,

$$\begin{aligned} T_i^{\mu_1 x_1} T_i^{\mu_2 x_2} \left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i \right) \right] &= T_i^{\mu_1 x_1} T_i^{\mu_2 x_2} \left[\left(\frac{1}{2} \sum_{k=1}^{n} g_{\mu_1 x_1 \ \mu x} T_k^{\mu x} \vec{I}_k \right) \times \left(\left(\frac{1}{2} \sum_{j=1}^{n} g_{\mu_2 x_2 \ \nu y} T_j^{\nu y} \vec{I}_j \right) \times \vec{I}_i \right) \right] \\ &= \frac{1}{4} \sum_{j,k} (\vec{I}_k \times (\vec{I}_j \times \vec{I}_i)) (T_i^{\mu_1 x_1} g_{\mu_1 x_1 \ \mu x} T_k^{\mu x}) (T_i^{\mu_2 x_2} g_{\mu_2 x_2 \ \nu y} T_j^{\nu y}) \\ &= \frac{1}{4} \sum_{j,k} (\vec{I}_k \times (\vec{I}_j \times \vec{I}_i)) L(i,k) L(i,j) = 0. \end{aligned} \tag{A.61}$$

Replace (A.60) and (A.61) into (A.59), we get

$$\sum_{i=1}^{n} \delta^3 (x - x_i) \left\{ T_i^{\mu_1 x_1} \left(\vec{a}_{\mu_1 x_1} \times \vec{I}_i \right) - T^{[\mu_1 x_1 \ \mu_2 x_2]} \left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i \right) \right] \right\} = 0, \tag{A.62}$$

which implies that:

$$T_i^{\mu_1 x_1} \left(\vec{a}_{\mu_1 x_1} \times \vec{I}_i \right) - T^{[\mu_1 x_1 \ \mu_2 x_2]} \left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i \right) \right] = 0. \tag{A.63}$$

Note that

$$
\begin{aligned}
T_i^{\mu_1 x_1}\left(\vec{a}_{\mu_1 x_1} \times \vec{I}_i\right) &= -\left(\vec{I}_i \times \vec{a}_{\mu_1 x_1}\right) T_i^{\mu_1 x_1} \\
&= -\left(\vec{I}_i \times \left(\frac{1}{2}\varepsilon^{\mu\nu\rho}\left(\vec{\lambda}_{\nu x} \times \vec{\lambda}_{\rho x}\right) g_{\mu x\ \mu_1 x_1} - \frac{1}{2}\sum_{j=1}^{n} T_j^{[\mu y\ \nu z]}\left(\vec{\lambda}_{\nu z} \times \vec{I}_j\right) g_{\mu y\ \mu_1 x_1}\right)\right) T_i^{\mu_1 x_1} \\
&= -\frac{1}{2}\varepsilon^{\mu\nu\rho}\vec{I}_i \times \left(\vec{\lambda}_{\nu x} \times \vec{\lambda}_{\rho x}\right) g_{\mu x\ \mu_1 x_1} T_i^{\mu_1 x_1} + \frac{1}{2}\sum_{j=1}^{n} T_j^{[\mu y\ \nu z]}\vec{I}_i \times \left(\vec{\lambda}_{\nu z} \times \vec{I}_j\right) g_{\mu y\ \mu_1 x_1} T_i^{\mu_1 x_1} \\
&= -\frac{1}{4}\sum_{j,k}\left[\vec{I}_i \times \left(\vec{I}_j \times \vec{I}_k\right)\right]\left(\frac{1}{2}\epsilon^{\mu\nu\rho} T_i^{\mu_1 x_1} T_j^{\mu_2 x_2} T_k^{\mu_3 x_3} g_{\mu x\ \mu_1 x_1} g_{\nu x\ \mu_2 x_2} g_{\rho x\ \mu_3 x_3}\right. \\
&\left. + T_i^{\mu_1 x_1} T_j^{[\mu y\ \nu z]} T_k^{\mu_2 x_2} g_{\mu_1 x_1\ \mu y} g_{\mu_2 x_2\ \nu z}\right),
\end{aligned}
\tag{A.64}
$$

$$
-T^{[\mu_1 x_1\ \mu_2 x_2]}\left[\vec{\lambda}_{\mu_1 x_1} \times \left(\vec{\lambda}_{\mu_2 x_2} \times \vec{I}_i\right)\right] = -\frac{1}{4}\sum_{j,k}\left[\vec{I}_j \times \left(\vec{I}_k \times \vec{I}_i\right)\right] T_j^{\mu_1 x_1} T_i^{[\mu y\ \nu z]} T_k^{\mu_2 x_2} g_{\mu_1 x_1\ \mu y} g_{\mu_2 x_2\ \nu z}.
\tag{A.65}
$$

Replacing (A.64) and (A.65) into (A.63),

$$
\begin{aligned}
&\sum_{j,k}\left[\left(\vec{I}_j \times \vec{I}_k\right) \times \vec{I}_i\right]\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho} T_i^{\mu_1 x_1} T_j^{\mu_2 x_2} T_k^{\mu_3 x_3} g_{\mu x\ \mu_1 x_1} g_{\nu x\ \mu_2 x_2} g_{\rho x\ \mu_3 x_3} + T_i^{\mu_1 x_1} T_j^{[\mu x\ \nu y]} T_k^{\mu_2 x_2} g_{\mu x\ \mu_1 x_1} g_{\nu y\ \mu_2 x_2}\right\} \\
&- \sum_{j,k}\left[\vec{I}_j \times \left(\vec{I}_k \times \vec{I}_i\right)\right] T_j^{\mu_1 x_1} T_i^{[\mu x\ \nu y]} T_k^{\mu_2 x_2} g_{\mu x\ \mu_1 x_1} g_{\nu y\ \mu_2 x_2} = 0,
\end{aligned}
\tag{A.66}
$$

and using the Latin index notation for discrete-continuous indices ($a \equiv \mu_1 x_1$), we have:

$$
\begin{aligned}
&\sum_{j,k}\left[\left(\vec{I}_j \times \vec{I}_k\right) \times \vec{I}_i\right]\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho} T_i^a T_j^b T_k^c g_{\mu x\ a} g_{\nu x\ b} g_{\rho x\ c} + T_i^a T_j^{[cd]} T_k^b g_{ca} g_{db}\right\} \\
&- \sum_{j,k}\left[\vec{I}_j \times \left(\vec{I}_k \times \vec{I}_i\right)\right] T_j^a T_i^{[cd]} T_k^b g_{ca} g_{db} = 0.
\end{aligned}
\tag{A.67}
$$

By taking the cross product of the previous equation with the current $\vec{I}_i$ and summing over the index i, we can write

$$
\begin{aligned}
&\sum_{i,j,k}\left[\left(\vec{I}_j \times \vec{I}_k\right) \times \vec{I}_i\right] \times \vec{I}_i\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho} T_i^a T_j^b T_k^c g_{\mu x\ a} g_{\nu x\ b} g_{\rho x\ c} + T_i^a T_j^{[cd]} T_k^b g_{ca} g_{db}\right\} \\
&- \sum_{j,k}\left[\vec{I}_j \times \left(\vec{I}_k \times \vec{I}_i\right)\right] \times \vec{I}_i\left\{T_j^a T_i^{[cd]} T_k^b g_{ca} g_{db}\right\} = 0.
\end{aligned}
\tag{A.68}
$$

Observe that

$$\begin{aligned}&\left[\left(\vec{I}_j\times\vec{I}_k\right)\times\vec{I}_i\right]\times\vec{I}_i=\left(\vec{I}_k\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_j\right)-\left(\vec{I}_j\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_k\right),\\&\left[\vec{I}_j\times\left(\vec{I}_k\times\vec{I}_i\right)\right]\times\vec{I}_i=\left(\vec{I}_k\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_j\right)\end{aligned}.$$

If we use these vector identities in the consistency equation (A.68), we have:

$$\begin{aligned}&\sum_{i,j,k}\left(\vec{I}_k\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_j\right)\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho}T_i^aT_j^bT_k^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}+T_i^aT_j^{[cd]}T_k^b g_{ca}g_{db}\right\}\\&-\sum_{i,j,k}\left(\vec{I}_j\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_k\right)\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho}T_i^aT_j^bT_k^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}+T_i^aT_j^{[cd]}T_k^b g_{ca}g_{db}\right\}\\&-\sum_{j,k}\left(\vec{I}_k\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_j\right)\left\{T_j^aT_i^{[cd]}T_k^b g_{ca}g_{db}\right\}=0,\end{aligned}\tag{A.69}$$

$$\begin{aligned}&\sum_{i,j,k}\left(\vec{I}_k\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_j\right)\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho}T_i^aT_j^bT_k^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}+(T_i^aT_j^{[cd]}T_k^b-T_j^aT_i^{[cd]}T_k^b)g_{ca}g_{db}\right\}\\&-\sum_{i,j,k}\left(\vec{I}_j\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_k\right)\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho}T_i^aT_j^bT_k^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}+T_i^aT_j^{[cd]}T_k^b g_{ca}g_{db}\right\}=0,\end{aligned}\tag{A.70}$$

and changing the indices $(j\leftrightarrow k)$ in the second term

$$\begin{aligned}&\sum_{i,j,k}\left(\vec{I}_k\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_j\right)\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho}T_i^aT_j^bT_k^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}+(T_i^aT_j^{[cd]}T_k^b-T_j^aT_i^{[cd]}T_k^b)g_{ca}g_{db}\right\}\\&-\sum_{i,j,k}\left(\vec{I}_k\times\vec{I}_i\right)\left(\vec{I}_i\cdot\vec{I}_j\right)\left\{\frac{1}{2}\varepsilon^{\mu\nu\rho}T_i^aT_k^bT_j^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}+T_i^aT_k^{[cd]}T_j^b g_{ca}g_{db}\right\}=0.\end{aligned}\tag{A.71}$$

Observe that

$$\begin{aligned}\varepsilon^{\mu\nu\rho}T_i^aT_k^bT_j^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}&=\varepsilon^{\mu\nu\rho}T_i^aT_k^bT_j^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}\quad\leftarrow(b\leftrightarrow c)\\&=\varepsilon^{\mu\nu\rho}T_i^aT_k^cT_j^b g_{\mu x\ a}g_{\nu x\ c}g_{\rho x\ b}\quad\leftarrow(\nu\leftrightarrow\rho)\\&=\varepsilon^{\mu\rho\nu}T_i^aT_k^cT_j^b g_{\mu x\ a}g_{\rho x\ c}g_{\nu x\ b}\quad\leftarrow(\varepsilon^{\mu\rho\nu}\leftrightarrow-\varepsilon^{\mu\nu\rho})\\&=-\varepsilon^{\mu\nu\rho}T_i^aT_j^bT_k^c g_{\mu x\ a}g_{\nu x\ b}g_{\rho x\ c}.\end{aligned}\tag{A.72}$$

If we combine this result with equation (A.71), we can write

$$\sum_{i,j,k}f_{i,j,k}\left[\varepsilon^{\mu\nu\rho}T_i^aT_j^bT_k^c g_{\mu xa}g_{\nu xb}g_{\rho xc}+\left(T_i^aT_j^{[cd]}T_k^b-T_j^aT_i^{[cd]}T_k^b-T_i^aT_k^{[cd]}T_j^b\right)g_{ca}g_{db}\right]=0,\tag{A.73}$$

and use the expression $D_{i\ a} = -T_i^b g_{ab}$:

$$\sum_{i,j,k} f_{i,j,k}\left[-\varepsilon^{\mu\nu\rho} D_{i\ \mu x} D_{j\ \nu x} D_{k\ \rho x} + \left(-T_i^{[cd]} D_{j\ c} D_{k\ d} + \underbrace{T_j^{[cd]} D_{i\ c} D_{k\ d}}_{(c\leftrightarrow d)} - T_k^{[cd]} D_{i\ c} D_{j\ d}\right)\right] = 0, \tag{A.74}$$

$$\sum_{i,j,k} f_{i,j,k}\left[-\varepsilon^{\mu\nu\rho} D_{i\ \mu x} D_{j\ \nu x} D_{k\ \rho x} + \left(-T_i^{[cd]} D_{j\ c} D_{k\ d} + T_j^{[dc]} D_{i\ d} D_{k\ c} - T_k^{[cd]} D_{i\ c} D_{j\ d}\right)\right] = 0, \tag{A.75}$$

$$\sum_{i,j,k} f_{i,j,k}\left[-\varepsilon^{\mu\nu\rho} D_{i\ \mu x} D_{j\ \nu x} D_{k\ \rho x} + \left(-T_i^{[cd]} D_{j\ c} D_{k\ d} - T_j^{[cd]} D_{i\ d} D_{k\ c} - T_k^{[cd]} D_{i\ c} D_{j\ d}\right)\right] = 0, \tag{A.76}$$

which can be rewritten using Latin letters for "discrete-continuous" indices, $c = \mu x$ y $d = \nu y$:

$$\begin{aligned}\sum_{i,j,k} f_{i,j,k}\Big[&- \varepsilon^{\mu\nu\rho} D_{i\ \mu x} D_{j\ \nu x} D_{k\ \rho x} \\ &- \left(T_i^{[\mu x\ \nu y]} D_{j\ \mu x} D_{k\ \nu y} + T_j^{[\mu x\ \nu y]} D_{i\ \nu y} D_{k\ \mu x} + T_k^{[\mu x\ \nu y]} D_{i\ \mu x} D_{j\ \nu y}\right)\Big] = 0,\end{aligned} \tag{A.77}$$

$$\sum_{i,j,k} 2 f_{i,j,k} \bar{\mu}(i,j,k) = 0 \ \Rightarrow \bar{\mu}(i,j,k) = 0, \tag{A.78}$$

where

$$\begin{aligned}\bar{\mu}(i,j,k) = -\frac{1}{2}\Big[&\varepsilon^{\mu\nu\rho} D_{i\ \mu x} D_{j\ \nu x} D_{k\ \rho x} \\ &+ \left(T_i^{[\mu x\ \nu y]} D_{j\ \mu x} D_{k\ \nu y} + T_j^{[\mu x\ \nu y]} D_{i\ \nu y} D_{k\ \mu x} + T_k^{[\mu x\ \nu y]} D_{i\ \mu x} D_{j\ \nu y}\right)\Big],\end{aligned} \tag{A.79}$$

is the TMC, which describes the linking of Borromean rings, and also vanishes. Therefore, we have demonstrated that our theory is consistent.