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Abelian path and signed-points space in the theories of Kalb-Ramond-Klein-Gordon and Schwinger

Enrique Alejandro Iñiguez Berrezueta

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Enrique Alejandro Iñiguez Berrezueta

Nombre del profesor, Titulo académico: Ernesto Contreras, PhD

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Nombres y apellidos:	Enrique Alejandro Iñiguez Berrezueta
Código:	00213564
Cédula de Identidad:	0924738164
Lugar y fecha:	May 20, 2024

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Resumen

En este trabajo de titulación, se exploran los espacios de caminos abelianos y puntos signados en las teorías de Kalb-Ramond-Klein-Gordon y Schwinger. Mediante el uso de representaciones geométricas, se analizan el generador de transformaciones duales en la teoría KRKG y la representación de monopolos magnéticos en la acción de Schwinger. Se revelan propiedades interesantes de cada modelo a través de un enfoque geométrico detallado.

Palabras clave: *Cuantización geométrica, teoría gauge Abeliana, monopolo magnético, transformación de dualidad*

Abstract

In this dissertation, we delve into the spaces of abelian paths and signed points within the Kalb-Ramond-Klein-Gordon and Schwinger theories. Through the use of geometric representations, we analyze the generator of dual transformations in the KRKG theory and the representation of magnetic monopoles in the Schwinger action. Interesting properties of each model are revealed through a detailed geometric approach.

Keywords: *Geometric quantization, Abelian gauge theory, magnetic monopole, duality transformation*

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Chapter 1

Introduction

In this chapter, we will set the basis for understanding the process of Canonical Quantization which we will use to formulate Quantum Field Theories of two different models: the free theory of the Kalb-Ramond and Klein-Gordon fields, and the theory of Schwinger for electromagnetism with sources in the presence of a magnetic monopole. We will go through the Hamiltonian formulation of singular Lagrangians and the quantization procedure. Finally, we will provide a brief statement of the models with which we will be working throughout this thesis and use the tools developed previously to proceed with the quantization of the theories.

1.1 Hamiltonian Formulation

The stationary action principle states that the dynamics of a mechanical system can be obtained by extremizing its action [1]. The natural representation of the action of a system is given by its Lagrangian:

$$S = \int dt L(q_i, \dot{q}_i), \tag{1.1}$$

which is a function of the generalized coordinates (q_i) and velocities ($\dot{q}_i$). Another equivalent formulation is given by the Hamiltonian, which is related to the Lagrangian by a Legendre transform:

$$p^i = \frac{\partial L}{\partial \dot{q}}, \tag{1.2}$$

$$H(q_i, p^i) = p^i \dot{q}_i(q_i.p^i) - L(q_i, \dot{q}_i(q_i.p^i)), \tag{1.3}$$

where p^i are called the conjugate momenta. It can be shown that the action depends only on the generalized coordinates and momenta (p^i). The equations of motion obtained by extremizing the action in the Hamiltonian formulation are:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p^i}, \\ \dot{p^i} &= -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{1.4}$$

We can define the mathematical construction of the Poisson brackets

$$\{A.B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial B}{\partial q_i}, \tag{1.5}$$

with which Hamilton's equations are written as:

$$\begin{aligned} \dot{q}_i &= \{q_i, H\}, \\ \dot{p}^i &= \{p^i, H\}. \end{aligned} \tag{1.6}$$

1.1.1 Hamiltonian formulation for finite degrees of freedom

The formulation with which we have proceeded presupposes that we can express the generalized velocities as functions of the coordinates and momenta. However, we may encounter relations of the form:

$$\phi(q_i, p^i) = 0, \tag{1.7}$$

which we will call primary constraints [2]. If this is the case, then we need a new formulation that will take into account the constraints found. This is done through the method of Lagrange multipliers. We can define a new Hamiltonian H^*:

$$H^* = H + \lambda^m \phi_m, \tag{1.8}$$

where ϕ_m are the primary constraints and λ^m are the Lagrange multipliers. With the new Hamiltonian defined in (1.8). Hamilton's equations are now taken to be:

$$\begin{aligned} \dot{q}_i &= \{q_i, H^*\}, \\ \dot{p}^i &= \{p^i, H^*\}, \\ \phi_m(q_i, p^i) &= 0. \end{aligned} \tag{1.9}$$

It must be noted that the constraints must not be set equal to zero until all Poisson brackets are calculated. To indicate this, we introduce the notation

$$\phi_m(q_i, p^i) \approx 0, \tag{1.10}$$

which reads as "weakly equal to", as first stated by Dirac [2]. Primary constraints must be preserved over time, so we demand:

$$\{\phi_m, H^*\} \approx \{\phi_m, H\} = 0. \tag{1.11}$$

Preservation of constraints can lead to one of four outcomes:

1. We get an equality $0 = 0$.
2. We find new constraints $\tilde{\phi}_v(q_i, p^i) \approx 0$ which we will call secondary constraints.
3. We find the Lagrange multipliers as functions of the canonical variables $\lambda^m = \lambda^m(q_i, p^i)$.
4. We get an inconsistency of the form $1 = 0$.

It is guaranteed that the process of preservation of constraints will not go on indefinitely and will end in 1., 3., or 4. If new constraints are found we repeat the preservation process until we get 1. or 3. Should we get 4. then the theory must be discarded. At the end of the process, we will end with $M+\tilde{M}$ constraints which we will simply call M. With these, we can construct an extended Hamiltonian by:

$$\tilde{H} = H + \lambda^m \phi_m. \tag{1.12}$$

Hamilton's equations can be obtained from an extended variational principle:

$$\begin{aligned} \dot{q}_i &= \{q_i, \tilde{H}\}, \\ \dot{p}^i &= \{p^i, \tilde{H}\}, \end{aligned} \tag{1.13}$$

where automatically $\phi_m \approx 0$ for all constraints. Time evolution for a quantity of phase space $g(q_i, p^i)$ will be given by

$$\dot{g} = \{g, \tilde{H}\}. \tag{1.14}$$

We note that the original Hamilton's equations cannot be recovered from the extended Hamiltonian. Nonetheless, both Hamiltonians describe the same physics, as we shall see. Thus, time preservation can and will be taken using the extended Hamiltonian from this point forward. From imposing the preservation of constraints over time we get:

$$\{\phi_m, H\} + \lambda^{m'}\{\phi_m, \phi_{m'}\} \approx 0. \tag{1.15}$$

This equation can be seen as a matrix equation, where $\{\phi_m, \phi_{m'}\}$ is an $M \times M'$ matrix and $\lambda^{m'}$ is a column vector. We can see that when it is not possible to find the Lagrange multipliers then they belong in the kernel of the matrix. Said subspace can be spanned by a basis of A vectors V_a^m, where A is the dimension of the kernel. Then, the extended Hamiltonian can be written as:

$$\tilde{H} = H + U^m \phi_m + v^a V_a^m \phi_m = H + U^m \phi_m + v^a \phi_a, \tag{1.16}$$

where $U^m \in \text{Col}\{\phi_m, \phi_{m'}\}$. We see that $V_m^a \phi_m$ projects the set of constraints on the kernel and the projected constraint is called ϕ_a.

We define a quantity whose Poisson bracket with all the set of constraints to be weakly equal to zero to be a **first class quantity**. On the contrary, we call **second class quantity** to any whose Poisson bracket with at least one constraint is not zero (not even weakly). The constraints that satisfy these definitions are named **first-class and second-class constraints** respectively [3].

As such, we see that there must be A first-class constraints and $R = M - A$ second-class constraints which we will call χ_r. The constraints can be rearranged so that:

$$\tilde{\Delta} = \begin{pmatrix} \{\chi_r, \chi_{r'}\} & \{\chi_r, \phi_{a'}\} \\ \{\phi_a, \chi_{r'}\} & \{\phi_a, \phi_{a'}\} \end{pmatrix} \approx \begin{pmatrix} \{\chi_r, \chi_{r'}\} & \mathbb{O}_{r\times a'} \\ \mathbb{O}_{a\times r'} & \mathbb{O}_{a\times a'} \end{pmatrix}, \tag{1.17}$$

is the rearranged constraints matrix. The blocks that include first-class constraints are zero from their definition. Dirac enunciated and proved a theorem that states that the determinant of the second-class constraints matrix is not zero, not even weakly [2]. Therefore, there must be an even number of second-class constraints.

One can define the Dirac brackets as:

$$\{A,B\}^{*}=\{A,B\}-\{A,\phi_m\}\tilde{\Delta}^{mm'}\{\phi_{m'},B\}, \tag{1.18}$$

where $\tilde{\Delta}^{mm'}$ are the elements of the inverse matrix of $\tilde{\Delta}$

Note that for first-class quantities the Dirac and Poisson brackets are equivalent. Similarly, observe that the Dirac bracket of any phase-space function with a second-class constraint is strongly zero. Time evolution is given by:

$$\dot{g}=\{g,H+v^a\phi_a\}^{*}. \tag{1.19}$$

This process allows us to deal with constrained systems in their Hamiltonian formulation. After we are done, we can quantize a theory with finite degrees of freedom as we shall later see. The next step will be to realize this procedure for systems with infinite degrees of freedom.

1.1.2 Hamiltonian formulation for infinite degrees of freedom

The formulation we have constructed so far is valid for finite degrees of freedom. For a system with infinite degrees of freedom, the Lagrangian and the Hamiltonian contain information on all the possible shapes of the fields at each point in space simultaneously. Thus, when discussing (classical or quantum) field theories it is useful to define a Lagrangian density $\mathscr{L}$ as:

$$L[\Phi, \dot{\Phi}] = \int d^3x \mathscr{L}(\Phi, \partial_\mu \Phi). \tag{1.20}$$

Analogous to the formulation for systems of finite degrees of freedom, the conjugate momenta and Hamiltonian density for a field theory are defined as:

$$\Pi^I = \frac{\delta L}{\delta \dot{\Phi}_I} = \frac{\partial \mathscr{L}}{\partial \dot{\Phi}_I}, \tag{1.21}$$

$$\mathscr{H} = (\Pi^I \dot{\Phi}_I - \mathscr{L})\Big|_{\Pi^I}, \tag{1.22}$$

where $\delta/\delta\Phi$ represents the functional derivative with respect to the field Φ [4]. This formulation, once again, presupposes that the "field velocities" can be obtained in terms of the momenta. When this is not the case, we resort to the constraints formulation discussed above with adequate changes: Poisson brackets are defined with functional derivatives, discrete indexes are changed to continuous variables, and summations are substituted by integration on the continuous variables.

With this formulation and with the tools introduced in the previous section we can proceed with the quantization of theories with finite and infinite degrees of freedom.

1.2 Canonical Quantization

Dirac's quantization procedure generates a quantum theory corresponding to a certain classical theory. It does so by reinterpreting dynamical variables as commutators that obey canonical commutation relations. The commutation algebra is inherited from the canonical algebra that these variables obey, generally, in terms of the Dirac brackets (recalling that these are equivalent to Poisson brackets if there are no second-class constraints). In the following discussion, we will consider the quantization of classical field theories, recalling that there exists a direct analogy between systems with finite and infinite degrees of freedom.

1.2.1 Quantization of systems without constraints

In the Schrodinger picture and natural units ($c = \hbar = 1$) [5]:

1. Dynamical variables are promoted to operators that act on elements of a Hilbert space:
$$\Phi_I(x^\mu), \Pi^I(x^\mu) \longrightarrow \hat{\Phi}_I(\vec{x}), \hat{\Pi}^I(\vec{x}),$$
where $I = 1, 2, 3, \ldots$ is the number of fields.

2. The commutator algebra is inherited from the algebra of Poisson brackets at equal times:
$$[\hat{\Phi}_I(\vec{x}), \hat{\Pi}^J(\vec{y})] = i\big\{\Phi_I(\vec{x}, t), \Pi^I(\vec{y}, t)\big\} = i\delta_I^J \delta^3(\vec{x} - \vec{y})$$

3. The dynamics of the quantum system will be indicated by vectors of a Hilbert space which evolve according to Schrodinger's equation:
$$\hat{H}\,|\psi\rangle = -i\frac{\partial}{\partial t}\,|\psi\rangle$$

Note that there is no restriction on how the operators act on the elements of the Hilbert space. A more specific assignment of the operators amounts to choosing a representation to work on.

1.2.2 Quantization of a system with only first-class constraints

In the Schrodinger Picture and natural units:

1. Dynamical variables are promoted to operators that act on elements of a Hilbert space:
$$\Phi_I(x^\mu), \Pi^I(x^\mu) \longrightarrow \hat{\Phi}_I(\vec{x}), \hat{\Pi}^I(\vec{x})$$
with $I = 1, 2, 3, \dots$

2. The commutator algebra is inherited from the algebra of Poisson brackets at equal times:
$$[\hat{\Phi}_I(\vec{x}), \hat{\Pi}^J(\vec{y})] = i\big\{\Phi_I(\vec{x}, t), \Pi^I(\vec{y}, t)\big\} = i\delta_I^J \delta^3(\vec{x} - \vec{y})$$

3. The dynamics of the quantum system will be indicated by vectors of a Hilbert space. The physical states are those that are destroyed by first-class constraints.
$$\hat{\phi}_a \left|\psi_{ph}\right\rangle = 0$$

4. Physical states evolve according to Schrodinger's equation:

$$\hat{H}\,|\psi_{ph}\rangle = i\frac{\partial}{\partial t}\,|\psi_{ph}\rangle$$

This shows that first-class constraints restrict the Hilbert space in which the theory is developed. Once again, there is no imposition on how the operators act on the physical states.

1.2.3 Quantization of a system with only second-class constraints

1. Dynamical variables are promoted to operators that act on elements of a Hilbert space:

$$\Phi_I(x^\mu), \Pi^I(x^\mu) \longrightarrow \hat{\Phi}_I(\vec{x}), \hat{\Pi}^I(\vec{x})$$

with $I = 1, 2, 3, \ldots$

2. The commutator algebra is inherited from the algebra of Dirac brackets at equal times:

$$[\hat{\Phi}_I(\vec{x}), \hat{\Pi}^J(\vec{y})] = i\big\{\Phi_I(\vec{x},t), \Pi^I(\vec{y},t)\big\}^*$$

3. Second-class constraints are strongly zero, so they are promoted to the zero operator:

$$\hat{\chi_r} = \hat{0}$$

4. The dynamics of the quantum system will be indicated by vectors of a Hilbert

space, which evolve according to Schrodinger's equation:

$$\hat{H}\left|\psi\right\rangle = i\frac{\partial}{\partial t}\left|\psi\right\rangle$$

Notice that second-class constraints do not restrict the Hilbert space since they must annihilate all states. The choice of Dirac brackets over Poisson brackets for the algebra stems from the fact that $\{\chi_r, \chi_s\} \neq 0$, not even weakly, whereas $\{\chi_r, \chi_s\}^* = 0$ [6].

1.3 Theory of the Kalb-Ramond-Klein-Gordon Fields

Our theory consists of a massive scalar field ϕ (0−form) and a completely antisymmetric massive tensor potential $A_{\mu\nu}$ (2−form) [7]. We expect the theory to be invariant with respect to duality transformations. In order to achieve this, the relation between the dimensions of the forms and the dimension of the spacetime D is given by:

$$p + q = D - 1, \tag{1.23}$$

as discussed in [8–14].With $p = 2, q = 0$ the theory lives in 3 spacetime dimensions $(2 + 1)$. The action is given by:

$$S = \int d^3x\left(\frac{1}{12}B_{\mu\nu\lambda}B^{\mu\nu\lambda} + \frac{m^2}{2}A_{\mu\nu}A^{\mu\nu} + \partial_\mu\phi\partial^\mu\phi + \frac{m^2}{2}\phi^2\right) \tag{1.24}$$

where

$$B_{\mu\nu\lambda} = \frac{1}{2}\partial_{[\mu}A_{\nu\lambda]} = \partial_\mu A_{\nu\lambda} + \partial_\nu A_{\lambda\mu} + \partial_\lambda A_{\mu\nu}, \tag{1.25}$$

we will now proceed with the process described in sections 1.1 and 1.2 to obtain a quantum field theory for the KRKG model.

1.3.1 Hamiltonian Formulation

The conjugate momenta for the canonical fields are:

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial_0 \phi = D \tag{1.26}$$

$$\Pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu\nu}} = F^{\mu\nu 0} = E^{\mu\nu}. \tag{1.27}$$

Now, we define:

$$\partial^i \phi = -\varepsilon^{ij} H^j \tag{1.28}$$

The Hamiltonian density $\mathcal{H}$ of the system can be written as:

$$\mathcal{H} = E^{ij}\dot{A}^{ij} - \frac{1}{4}E^{ij}E^{ij} + \frac{1}{2m^2}\partial_i E^{ik}\partial_j E^{jk} + \frac{m^2}{4}A_{ij}A^{ij} + \frac{1}{2}D \cdot D + \frac{1}{2}H_i H^i + \frac{m^2}{2}\phi^2 \tag{1.29}$$

Immediately we get the primary constraint:

$$\Pi^{0\nu} = E^{0\nu} = 0 \tag{1.30}$$

Preserving this constraint in time we get:

$$\dot{\Pi}^{0\nu} = \partial_i E^{ij} + m^2 A^{j0} = 0 \tag{1.31}$$

These are second-class constraints. Dirac brackets are written as:

$$\begin{aligned}
\{A_{ij}(\vec{x}), E^{mn}(\vec{y})\}^* &= \delta_{ij}^{mn}\delta^2(\vec{x}-\vec{y}) \\
\{\phi(\vec{x}), D(\vec{y})\}^* &= \delta^2(\vec{x}-\vec{y}) \\
\{A_{ij}(\vec{x}), A_{mn}(\vec{y})\}^* &= \{E^{ij}(\vec{x}), E^{mn}(\vec{y})\}^* = 0 \\
\{\phi(\vec{x}), \phi(\vec{y})\}^* &= \{D(\vec{x}), D(\vec{y})\}^* = 0
\end{aligned} \tag{1.32}$$

This concludes the Hamiltonian formulation of the KRKG model. From here it is possible to obtain the equations of motion for the fields and quantize the theory, which will be done in the following sections.

1.3.2 Equations of motion and duality transformations

Taking variations on the action we get the following equations of motion:

$$\partial_\lambda B^{\mu\nu\lambda} + m^2 A^{\mu\nu} = 0 \tag{1.33}$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{1.34}$$

which are the equations for the massive Kalb-Ramond Field and the Klein-Gordon equation for the scalar field.

With equations (1.26) to (1.28), the equations for the KR and KG fields are summarized in the following equations:

$$\partial_i E^{ij} + m^2 A^{j0} = 0 \tag{1.35}$$

$$\partial_0 E^{ij} + m^2 A^{ij} = 0 \tag{1.36}$$

$$E^{ij} - \partial_i A^{j0} - \partial_j A^{0i} - \partial_0 A^{ij} = 0 \tag{1.37}$$

$$D + \partial_0 \phi = 0 \tag{1.38}$$

$$\partial_i \phi + \varepsilon^{ij} H^j = 0 \tag{1.39}$$

$$\partial_0 D + \varepsilon^{ij} \partial_i H^j + m^2 \phi = 0 \tag{1.40}$$

Notice that three of these are the definitions we made for E^{ij}, D y H^i. It can also be seen that the generalized magnetic field would be defined as:

$$B = \frac{1}{2}\varepsilon^{ij} E^{ij} = E$$

which is a scalar, but provides redundant information already provided by the electric field tensor. It is possible to map both fields in three spacetime dimensions with a duality transformation, since $p = 2, q = 0$ satsify (1.23) as shown in [15]. Additionally, since p and q are both even the duality is continuous, given by $SO(2)$.

If we define the objects:

$$\tilde{F} = \begin{pmatrix} E^{ij} \\ D \\ H^i \end{pmatrix}, \tag{1.41}$$

$$\tilde{A} = \begin{pmatrix} m\varepsilon^{ij}1\phi \\ \frac{m}{2}\varepsilon^{ij}A^{ij} \\ -mA^{i0} \end{pmatrix}, \tag{1.42}$$

then the transformation is given by

$$\begin{pmatrix} \tilde{F}' \\ \tilde{A}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{F} \\ \tilde{A} \end{pmatrix}. \tag{1.43}$$

It can be seen that this transformation preserves the equations of motion and keeps the action invariant. Furthermore, this duality transformation can be implemented infinitesimally, which, by Noether's theorem implies the existence of a quantity G that is preserved in time [3]. In the following sections, we will explicitly calculate this quantity which we will call the generator of the transformation and promote its to a quantum operator to extract interesting results.

1.3.3 Generator of duality of the Kalb-Ramond-Klein-Gordon model

The duality transformation described in the previous section can be implemented infinitesimally for the canonical variables A_{ij}, E^{ij}, ϕ, D as:

$$\begin{aligned} \delta E^{ij} &= \theta m \varepsilon^{ij} \phi \\ \delta D &= \theta \frac{m}{2} \varepsilon^{ij} A^{ij} \\ \delta \phi &= -\frac{\theta}{2m} \varepsilon^{ij} E^{ij} \\ \delta A^{ij} &= \frac{\theta}{2m} \varepsilon^{ij} D \end{aligned}$$

which, as a consequence of Noether's Theorem, implies there must exist a quantity conserved over time [3]. Furthermore, said quantity G happens to satisfy:

$$\{\theta G, \varphi\} = \theta \delta \varphi, \tag{1.44}$$

$$\{\theta G, \Pi\} = \theta \delta \Pi. \tag{1.45}$$

Thus, we call this quantity the generator of the transformation. We can find the generator of a transformation by considering the following: say that the fields change infinitesimally as $\delta \Phi_I = g_I(\theta)\delta\theta$ then the effect of the transformation on the action is to change it by a total time derivative:

$$\delta_\theta S = \int d^4x \, \frac{\mathrm{d}F}{\mathrm{d}t} \tag{1.46}$$

If equations of motion hold then:

$$\delta S = \int d^4x \left(\frac{\mathrm{d}}{\mathrm{d}t}(\Pi\delta\Phi) + \left(\dot{\Pi} + \frac{\partial \mathcal{H}}{\partial \Phi} + \partial_i \frac{\partial \mathcal{H}}{\partial(\partial_i \Phi)} \right) \delta\phi + \left(\dot{\Phi} - \frac{\partial \mathcal{H}}{\partial \Pi} \right) \delta\Pi \right) = \int d^4x \frac{\mathrm{d}}{\mathrm{d}t}(\Pi\delta\Phi) \tag{1.47}$$

The generator will then be the time-preserved quantity:

$$G = \int d^4x (F - \Pi\delta\Phi) \tag{1.48}$$

For the transformation described in (1.43) the generator is given by:

$$G = \int d^3x \left(\frac{m}{2} \varepsilon^{ij} \phi A_{ij} + \frac{1}{2m} \varepsilon^{ij} D E^{ij} \right) \tag{1.49}$$

We can now use the process described in section 1.3.4 to represent the generator in equation (1.49). The following section will deal with the quantization for the KRKG model.

1.3.4 Canonical quantization

We have a theory with only second-class constraints given by the Hamiltonian in (1.29). We thus proceed with the Dirac Method for quantizing the model:

1. Canonical variables are promoted to operators that act on elements of a

Hilbert space:

$$A_{ij} \longrightarrow \hat{A}_{ij}$$

$$E^{ij} \longrightarrow \hat{E}^{ij}$$

$$\phi \longrightarrow \hat{\phi}$$

$$D \longrightarrow \hat{D}$$

2. The commutator algebra is inherited from the Dirac (Poisson) brackets in (1.32).

3. The dynamics are dictated by the Schröndiger equation:

$$\hat{H}\left|\psi\right\rangle = i\frac{\partial}{\partial t}\left|\psi\right\rangle$$

with the Hamiltonian as defined in (1.29)

We now have a quantum field theory for the KRKG model. The next step is to choose how the operators act upon the wave functions and obtain the quantum representation of the generator we found in the previous section. Nonetheless, we shall take a detour to study the theory of Schwinger for electromagnetism with sources. We will come back to this result after developing the machinery of geometrical representation for a more insightful analysis.

1.4 Schwinger Theory of Electromagnetism with monopoles

Maxwell's equations for electromagnetism with sources are summarized in:

$$\partial_\nu F^{\mu\nu} = J_e^\mu,$$
$$\partial_\nu \star F^{\mu\nu} = 0.$$

This set of equations is not symmetrical which reflects that they do not consider the existence of magnetic charges or currents. Dirac found that if magnetic charges existed, then the electric charge must be quantized as:

$$\frac{eg}{4\pi} = \frac{1}{2}n,\; n \in \mathbb{Z}, \tag{1.50}$$

where e is the quantum of electric charge and g is the unit of magnetic charge [16]. Schwinger found that to align with experimental results, in which only electric charges are found and are significantly small, then large magnetic charges have not been produced [17]. It is of interest to analyze a theory of electromagnetism that considers both electric and magnetic charges. Such a theory can be characterized by the Schwinger action in $3+1$ dimensions [17]:

$$S = \int d^4x \left(A_\mu J_e^\mu + B_\mu J_m^\mu - \frac{1}{2} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right), \tag{1.51}$$

where B_μ is defined as:

$$B_\mu(x) = \int d^4y \star F_{\mu\nu}(y) f^\nu(y-x) + \partial_\mu \lambda(x). \tag{1.52}$$

In this expression, λ is an arbitrary function and $f(x)$ obeys:

$$\partial_\mu f^\mu(x) = \delta^4(x), \tag{1.53}$$

where $\star$ is the Hodge Dual operator.

Unlike Maxwell's electromagnetism, note that we can assume that $A_\mu, F_{\mu\nu}$ are not related through an exterior derivative and take independent variations with respect to both fields. Taking variations with respect to A_μ yields the equations:

$$\partial_\nu F^{\mu\nu} = J_e^\mu, \tag{1.54}$$

which we recognize from Maxwell's Electromagnetism. Now, variations with respect to $F_{\mu\nu}$ yield:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} \int d^4y\, J_m^\alpha(y) f^\beta(x-y). \tag{1.55}$$

Taking the dual of this expression we get:

$$\star F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}(\partial^\sigma A^\rho - \partial^\rho A^\sigma) + \int d^4y\, J_m^\mu(y) f^\nu(x-y), \tag{1.56}$$

which can be alternatively expressed as:

$$\partial_\nu^x \star F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}\partial_\nu^x(\partial^\sigma A^\rho + \partial^\rho A^\sigma) + J_m^\sigma(x) = J_m^\mu(x), \tag{1.57}$$

which differs from Maxwell's Electromagnetism by the existence of a magnetic current density. If we take $J_m = 0$ then we should recover classical electromagnetism. Another way to express equation (1.55) is:

$$A_\mu(x) = -\int d^4y\, f^\nu(x-y)F_{\mu\nu}(y) + \partial_\mu\lambda_e(x), \tag{1.58}$$

which is analogous to (1.52).

There exists a symmetry in the equations of motion under duality rotations characterized by:

$$\begin{pmatrix} \mathcal{A}' \\ \mathcal{B}' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}, \tag{1.59}$$

where $\mathcal{A} = \begin{pmatrix} J_e \\ F \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} J_m \\ \star F \end{pmatrix}$, provided that $f^\nu(x) = -f^\nu(-x)$.

To focus solely on the information from one monopole we take the other current density to be zero. In the case of the magnetic monopole, we take the magnetic current density to be

$$J_m^\mu = g\delta_0^\mu\delta^3(\vec{0}). \tag{1.60}$$

A convenient choice for f^ν is

$$f^\nu(x) = -\frac{1}{4\pi}\frac{x^i}{|\vec{x}|^3}\delta_i^\nu\delta(x_0), \tag{1.61}$$

that satisfies:

$$\partial_\nu f^\nu = \frac{1}{4\pi}\delta(x_0)\partial_i\frac{x^i}{|\vec{x}|^3} = \delta(x_0)\delta(\vec{x}) = \delta^4(x).$$

1.4.1 Hamiltonian Formulation

We seek to obtain a quantum field theory from the Schwinger model. Therefore, we start by developing the Hamiltonian formulation for the theory. We begin by taking the conjugate momenta:

$$\Pi^\mu \approx F^{i0}\delta_i^\mu \tag{1.62}$$

$$\Pi^{\mu\nu} \approx 0 \tag{1.63}$$

Notice that we have then three canonical fields (A^0, A^i, Π^{ij}) and three canonical momenta (Π^0, Π^i, Π^{ij}) since we choose to treat F^{i0} as a momentum instead of a field. The Hamiltonian of the theory is given by:

$$H = \int d^3x\left(\frac{1}{2}\Pi^i\Pi^i - \frac{1}{4}F_{ij}^2 + \frac{1}{2}F_{ij}(f_{ij} - b_{ij}) + \partial_i\Pi^i A_0\right), \tag{1.64}$$

where we have defined:

$$f_{ij} = \partial_i A_j - \partial_j A_i, \tag{1.65}$$

$$b_{ij} = g\varepsilon_{ijk} f^k. \tag{1.66}$$

(1.62) with time index and (1.63) with spatial indices are primary constraints in the sense of Dirac. Time preservation of the constraints leads to:

$$\dot{\Pi}^0 = \partial_i \Pi^i \approx 0, \tag{1.67}$$

$$\dot{\Pi}^{ij} = F_{ij} - f_{ij} + b_{ij} \approx 0. \tag{1.68}$$

The constraint in equation (1.67) yields no further constraints, whereas equation (1.68) allows to find the Lagrange multipliers for the Π^{ij}. With this, we can express the total Hamiltonian as:

$$H^\star = H + \int d^3x \bigg(\lambda_0(x)\Pi^0 + \frac{1}{2}(\partial_j \Pi^i - \partial_i \Pi^j)\Pi^{ij} \bigg). \tag{1.69}$$

Constraints in (1.62) and (1.67) are first-class, while constraints in (1.63) and (1.68) are second-class. We can fix the gauge by setting $A_0 = 0$, with which this and (1.62) become second-class constraints. The only remaining first-class constraint is $\partial_i \Pi^i$. We find the Dirac brackets for the second-class constraints by noticing:

$$\left\{A_0(\vec{x}), \Pi^0(\vec{y})\right\} = \delta^3(\vec{x} - \vec{y}), \tag{1.70}$$

$$\left\{A_0, \Pi^{ij}\right\} = 0, \tag{1.71}$$

$$\{A_0, F_{ij} - f_{ij} + b_{ij}\} = 0, \tag{1.72}$$

$$\left\{\Pi^0, \Pi^{ij}\right\} = 0, \tag{1.73}$$

$$\left\{\Pi^0, F_{ij} - f_{ij} + b_{ij}\right\} = 0, \tag{1.74}$$

$$\left\{\Pi^{ij}(\vec{x}), F_{ij}(\vec{y}) - f_{ij}(\vec{y}) + b_{ij}(\vec{y})\right\} = -\frac{1}{2}\delta^3(\vec{x} - \vec{y}). \tag{1.75}$$

Construction of Dirac brackets show that:

$$\left\{A_i(\vec{x}), \Pi^i(\vec{y})\right\}^\star = \left\{A_i(\vec{x}), \Pi^i(\vec{y})\right\} = \delta^3(\vec{x} - \vec{y}), \tag{1.76}$$

whereas for all other canonical variables, Dirac brackets are immediately equal to their Poisson brackets as they are first-class. Furthermore, constraints in (1.62), (1.63), (1.68), and the gauge-fixing of A_0 can be set to be strongly zero since they are second-class, and thus the dynamics of the system is given by:

$$H = \int d^3x \left(\frac{1}{2}\Pi^i\Pi^i + \frac{1}{4}(f_{ij} - b_{ij})^2\right). \tag{1.77}$$

Note that for the case of $g = 0$ the Hamiltonian reduces to the Free Maxwell Hamiltonian.

1.4.2 Canonical Quantization of the Theory

The only constraint that remains is first class, so we proceed with the quantization of the theory *a la Dirac* for the remaining canonical variables A_i, Π^i.

1. Canonical variables are promoted to operators that act on elements of a Hilbert space.

$$\begin{aligned} A_i &\longrightarrow \hat{A}_i, \\ \Pi_i &\longrightarrow \hat{\Pi}_i. \end{aligned}$$

2. The commutator algebra is inherited from the Dirac (Poisson) brackets.

$$\begin{aligned} \left[\hat{A}_i(\vec{x}), \hat{A}_j(\vec{y})\right] &= 0, \\ \left[\hat{\Pi}^i(\vec{x}), \hat{\Pi}^j(\vec{y})\right] &= 0, \\ \left[\hat{A}_i(\vec{x}), \hat{\Pi}^j(\vec{y})\right] &= \delta_i^j \delta^3(\vec{x}-\vec{y}). \end{aligned} \tag{1.78}$$

3. The physical section of the Hilbert space is that which is annihilated by the primary constraint:

$$\partial_i \hat{\Pi}^i \left|\psi\right\rangle = 0.$$

4. On the physical section, the dynamics are dictated by the Schröndiger equation:

$$\hat{H} \left|\psi\right\rangle = i \frac{\partial}{\partial t} \left|\psi\right\rangle .$$

with the Hamiltonian given by:

$$\hat{H} = \int d^3 \left(\hat{\Pi}^i \hat{\Pi}^i + \frac{1}{4} (\partial_i \hat{A}_j - \partial_j \hat{A}_i - b_{ij}) \right). \tag{1.79}$$

This concludes our review of the KRKG and Schwinger models. In this chapter, we have provided an overview of the two models. We have obtained the Hamiltonian formulation for both theories and derived their corresponding equations of motion. On one hand, for the KRKG model we have explicitly calculated the generator of duality transformations obtained from Noether's theorem. On the other hand, we have found an expression for the Hamiltonian of the theory and compared it to the Hamiltonian of Maxwell's Electromagnetism. We finally carried out the canonical quantization procedure for both theories accounting for their constraints and their effects on the physical sector of the theory and the commutator algebra. In the subsequent chapter, we will introduce the geometric representation, which leverages abelian path space and its generalizations. This novel approach will offer new insights when promoting canonical fields into operators that act on wave functionals that depend on geometrical objects, thereby facilitating a comprehensive analysis of their quantum properties.

Chapter 2

Geometrical representation

In this chapter, we introduce the formalism of Abelian path space and its generalizations. We will construct the groups as a set of equivalence classes of topological objects with a certain group operation and we will develop operators such as functional derivatives that act on functionals that depend on elements of these groups. This is the framework that we will use in the realization of the quantum operators in our models in what we label as geometric representation.

2.1 Path-space representation

2.1.1 Construction of the Abelian path space

The geometric representation works with operators that act on curves, i.e. 1-surfaces. We define the space of curves P as the set of all 1-surfaces on a manifold M; that is, all maps $\gamma : [s_1, s_2] \cup \cdots \cup [s_{n-1}, s_n] \longrightarrow M$ that are smooth on each interval $[s_i, s_{i+1}]$ and continuous over the entire domain [18]. Without loss of generality, we can take $M = \mathbb{R}^n$. The composition of paths is defined as the map:

$$\gamma_1 \circ \gamma_2 = \begin{cases} \gamma_1(2s) & s \in [0, \frac{1}{2}] \\ \gamma_2(2(s - \frac{1}{2})) & s \in [\frac{1}{2}, 1] \end{cases} \tag{2.1}$$

In other words, it is the path that first traverses the curve γ_1 and then the curve γ_2. We are interested in the geometric objects rather than any parametrizations of the curves. We define equivalence classes given by:

$$\gamma_1 \sim \gamma_2 \iff \exists \varphi : D \longrightarrow D \mid \gamma_1 \circ \varphi = \gamma_2. \tag{2.2}$$

In short, any two parametrizations of one curve are equivalent. Nonetheless, this set still does not have a group structure. Then, to endow the set with a group structure, a second equivalence class is defined by:

$$T^i(\vec{x}, \gamma) = \int_\gamma dy^i \, \delta^n(\vec{x} - \vec{y}).$$

This object is denoted as the form factor of a curve. We say that two curves are equivalent if they have the same form factor, i.e., $\gamma_1 \sim \gamma_2 \iff T^i(\vec{x}, \gamma_1) = T^i(\vec{x}, \gamma_2)$. In taking equivalence classes for the form factor of the curves all parametrizations of a single curve already belong in the same equivalence class. Note that the form factor of the curve indicates the distribution of tangent vectors along the curve [18]. The equivalence classes defined by this form factor form an Abelian group under the composition of paths. This group is referred to as the Abelian path-space. If allowed by the manifold's dimension, it is possible to define a subgroup of the Abelian path space: the loop space, given by all equivalence classes on closed curves [15].

2.1.2 Path and Loop Derivatives

Since geometric quantization works with path-dependent functionals, it is reasonable to define some notion of differentiation. The path (loop) derivative $\delta_i(\vec{x})$ ($\Delta_{ij}(\vec{x})$) is defined as the change in the path-dependent functional as an infinitesimal path (loop) is attached to its argument γ at the point x. That is, up to first order:

$$u^i \delta_i(\vec{x})\psi[\gamma] = \psi[\gamma \circ u_x] - \psi[\gamma], \tag{2.3}$$

$$\frac{1}{2}\sigma^{ij}\Delta_{ij}(\vec{x})\psi[\gamma] = \psi[\gamma \circ \delta c] - \psi[\gamma]. \tag{2.4}$$

It can be shown that the two derivatives are connected by

$$\partial_i \delta_j(\vec{x}) - \partial_j \delta_i(\vec{x}) = \Delta_{ij}(\vec{x}). \tag{2.5}$$

Notice that the loop derivative is of dimension 2, since the loop encloses a 2-surface. This derivative can also be thought of in terms of the surfaces enclosed by the loops in the sense of the Stokes Theorem. [19]

To better understand how these operators act on a path-dependent functional we can take the path and loop derivatives of the form factor. First, we have that

$$T^i(\vec{x}, \gamma \circ u_y) = \int_{\gamma \circ u_y} dy^i \delta^n(\vec{x} - \vec{y}) = T(\vec{x}, \gamma) + u^j \delta^i_j \delta^n(\vec{x} - \vec{y}),$$

with which we get

$$\delta_i(\vec{y}) T^j(\vec{x}, \gamma) = \delta^j_i \delta^n(\vec{y} - \vec{y}). \tag{2.6}$$

Similarly for the loop derivative:

$$\Delta_{ij}(\vec{y}) T^k(\vec{x}, \gamma) = \delta^k_j \partial_i \delta^n(\vec{x} - \vec{y}) - \delta^k_i \partial_j \delta^n(\vec{x} - \vec{y}). \tag{2.7}$$

We now have a set of useful tools to analyze theories with $1-$form fields, such as Maxwell's and Schwinger's electromagnetism. However, it will be useful to extend this machinery to work with general $p-$form fields. An extension for $2-$forms will be useful for understanding the Schwinger model and the KRKG model. Additionally for the KRKG model, we need to understand the case for $0-$form fields; this extension will come from the formalism of the space of signed points. The next section will cover the extension of the tools developed in this chapter for general

$p-$form fields.

2.2 Extensions of path-space representation

The notion of the Abelian path space construction can be extended to more general $p-$surfaces. For example, for $p = 2$ we have the set of 2-surfaces in $\mathbb{R}^n$. Similarly, for the $p = 0$ case, the space we consider is the space of signed points, composed of all unordered lists of points on the manifold with a sign s_a $(\pm)$ assigned at each. Both generalizations are useful for analyzing more general $p-$form theories in their geometrical representation, such as the scalar and tensor fields.

2.2.1 The space of $p-$surfaces

The Abelian path-space construction can be generalized for $p-$form spaces [15]. In these cases, the objects that comprise the space are $p-$surfaces over a manifold M. We define the space of oriented $p-$surfaces S_p as all the maps Σ : $[s_1^1, s_2^1] \times \cdots \times [s_1^p, s_2^p] \bigcup \ldots [s_{n-1}^1, s_n^1] \times \cdots \times [s_{n-1}^p, s_n^p] \longrightarrow M$ that are smooth on each interval $[s_i, s_{i+1}]$ and continuous over the entire domain. The manifold can be chosen to be $\mathbb{R}^n$ directly, noting that the extension to an arbitrary manifold M is straightforward. The composition of surfaces follows the construction in (2.1). Since we are interested in the geometric objects rather than any particular parametrizations they may have, we define equivalence classes by:

$$\Sigma_1 \sim \Sigma_2 \iff \exists \vartheta : D \longrightarrow D | \Sigma_1 \circ \vartheta = \Sigma_2. \tag{2.8}$$

That is, any two parametrizations of the same $p-$surface are equivalent. The set of these equivalence classes with the composition of $p-$surfaces does not yet define a group. It is of interest to us to endow the set with a group structure. Following the process on 2.1.1 we define the form factor of a surface as:

$$T^{i_1 \dots i_p}(\vec{x}, \Sigma) = \int_\Sigma d\Sigma_{\vec{y}}^{i_1 \dots i_p} \delta^{D-1}(\vec{x} - \vec{y}). \tag{2.9}$$

We say that two $p-$surfaces are equivalent if their form factors coincide, which allows us to define new equivalence classes that with the composition of surfaces as previously defined, form an Abelian group. In this case, note that the form factor gives a measure of the distribution of normal vectors over the hypervolume. We will refer to this group as the Abelian $p-$surface space. If allowed by the manifold's dimension, it is possible to define a subgroup consisting of all the equivalence classes on closed $p-$surfaces, analogous to the loop space.

Both derivatives defined in 2.1.2 can be extended in the $p-$surface space to the open $p-$surface derivative $\delta_{i_1 \cdots i_p}(\vec{x})$ and the closed p-surface derivative $\Delta_{i_1 \cdots i_{p+1}}(\vec{x})$. The definitions for these are analogous to the 1-dimensional case:

$$\sigma^{i_1 \dots i_p} \delta_{i_1 \dots i_p}(\vec{x}) \psi[\Sigma] = \psi[\Sigma \circ \sigma_x] - \psi[\Sigma], \tag{2.10}$$

$$\frac{1}{(p+1)!} \sigma^{i_1 \dots i_{p+1}} \Delta_{i_1 \dots i_{p+1}}(\vec{x}) \psi[\Sigma] = \psi[\Sigma \circ \delta\sigma] - \psi[\Sigma]. \tag{2.11}$$

Thus, for $p = 2$ the open (closed) derivative is taken as the change in the 2-surface-dependent functional as an infinitesimal open (closed) surface is attached to its argument Σ at the point x. Notice as well that the closed $p-$surface derivative is of dimension $p+1$. Both derivatives are related by:

$$\Delta_{ii_1\dots i_{p+1}}(\vec{x}) = \frac{1}{p!}\partial_{[i}\delta_{i_1\dots i_p]}. \tag{2.12}$$

To exemplify how these derivatives act on $p-$surface dependent functionals, let us calculate the open and closed $p-$surface derivatives of the form factor for the $p=2$ case:

$$T^{ij}(\vec{x},\Sigma\circ\sigma_{\vec{y}}) = \int\limits_{\Sigma\circ\sigma_{\vec{y}}} d\Sigma^{ij}_{\vec{y}}\delta^{D-1}(\vec{x}-\vec{y}) = T^{ij}(\vec{x},\Sigma) + \sigma^{ij}\delta^{D-1}(\vec{x}-\vec{y}),$$

from where

$$\delta_{kl}(\vec{y})T^{ij}(\vec{x},\Sigma) = \delta^{ij}_{kl}\delta^{D-1}(\vec{x}-\vec{y}), \tag{2.13}$$

and for the closed surface derivative:

$$\Delta_{ijk}(\vec{x})T^{lm}(\vec{x},\Sigma) = (\partial_i\delta^{lm}_{jk} + \partial_j\delta^{lm}_{ki} + \partial_k\delta^{lm}_{ij})\delta^{D-1}(\vec{x}-\vec{y}). \tag{2.14}$$

2.2.2 The space of signed points

For the $p=0$ case we turn to the construction of the space of signed points, as discussed in [20]. First, consider the set L of all ordered lists X over a manifold M whose elements are the pairs $X_a = (x_a, s_a)$, where $x_a \in M$ and $s_a = \pm 1$. These lists can be represented more compactly by:

$$X = \{x_1^{(s_1)}, \dots, x_n^{(s_n)}\}. \tag{2.15}$$

The elements x_a of the lists are defined as signed points. It is seen that there are two distinct types of elements: those plus-signed, which will be referred to as points, and those minus-signed, which will be referred to as antipoints. The number of elements in each list is arbitrary. The composition of lists is defined as:

$$XY = \{x_1^{(s_1)}, \ldots, x_n^{(s_n)}, y_1^{(r_1)}, \ldots, y_m^{(r_m)}\}, \tag{2.16}$$

where $X, Y \in L$ are lists of n and m points respectively. It is interesting to endow this set with a group structure using list composition. To do this we induce a map $R : L \longrightarrow L$ defined by

$$R(\{x_a^{(+)}, x_a^{(-)}\}) = \{\}, \tag{2.17}$$

where $\{\}$ is the empty list, i.e., the list with no points. R defines equivalence classes $[X]$ on L, given by:

$$X \sim Y \iff R(X) = R(Y) \tag{2.18}$$

The set of all equivalence classes L_R, along with the composition of lists defined above has a group structure. It is of further interest to consider an Abelian version of the group of reduced lists just defined. For this, consider the following function $T : M \times L_R \longrightarrow \mathbb{Z}$:

$$T(\vec{x}, X) = \sum_a s_a \delta^n(\vec{x} - \vec{x_a}). \tag{2.19}$$

This map is denoted as the form factor of a list, and it simply extracts the sign-factor s_a of each element X_a of a list. Note that it is an analogous construction to the form factor defined for the $p-$surface space in 2.2.1. The form factor defines new equivalence classes $[X]$ on the set of reduced lists given by the rule

$$X \sim Y \iff T(\vec{x}, X) = T(\vec{x}, Y).$$

Moreover, it can be seen, using the definitions of section 2.1.1 that

$$T(\vec{x}, XY) = T(\vec{x}, X) + T(\vec{x}, Y), \tag{2.20}$$

which gives that $[X][Y] = [XY]$, and endows the group of equivalence classes given by the form factor an Abelian structure. These new equivalence classes relax the order condition that a point and antipoint must be contiguous to cancel. Mathematically, this is equivalent to redefining the map R as:

$$R(X_1\{x_a^{(+)}\}X_2\{x_a^{(-)}\}X_3) = X_1X_2X_3. \tag{2.21}$$

Analogous to the process defined on 2.2.1, a "dipole derivative" can be defined which measures the change in the list-dependent functional when an infinitesimal list of points i.e., a list of two points with different signs, $\vec{x}^{(+)}$ and $(\vec{x} + \vec{\delta u})^{(-)}$ in the limit where $\left\|\vec{\delta u}\right\| \to 0$, is added to its argument. That is:

$$u^\mu \Delta_\mu(\vec{x})\psi[X] = \psi[X\{\vec{x}^{(+)}, (\vec{x} + \vec{\delta u})^{(-)}\}] - \psi[X]. \tag{2.22}$$

The change is taken up to first order. This derivative is analogous to the loop derivative defined for $p-$surfaces. For the sake of being explicit, we calculate the dipole derivative for the form-factor:

$$T(\vec{y}, X\delta X) = T(\vec{y}, X) + \delta^2(\vec{x} + \vec{u} - \vec{y}) - \delta^2(\vec{x} - \vec{y}) = T(\vec{y}, X) + u^\mu \frac{\partial}{\partial x^\mu}\delta^2(\vec{x} - \vec{y}). \tag{2.23}$$

thus we get:

$$\Delta_\mu(\vec{x})T(\vec{y}, X) = \frac{\partial}{\partial x^\mu}\delta^2(\vec{x} - \vec{y}). \tag{2.24}$$

We have now developed all the tools we need to analyze the two models presented in chapter 1. In the following section we proceed with the promotion of the canonical variables to operators that act on $p-$surfaces, namely for $p = 0, 1,$ and 2. We shall see the results that arise from analysing the KRKG and the Schwinger models in this geometrical representation.

Chapter 3

Results

Applying the machinery of Chapter 2 we now turn again to our two models in discussion: The generator of duality transformations in the KRKG theory, and the representation of magnetic monopoles in the Schwinger action. Geometrical representation will provide insightful analysis for both models and reveal different interesting properties of each.

3.1 Generator of duality transformations in the geometric representation

We will choose to project our Hilbert space on the product space of surface-dependent and list-dependent functionals:

$$\psi[\Sigma; X] = \langle \Sigma; X|\psi\rangle \,, \tag{3.1}$$

with $|\Sigma; X\rangle \in |\Sigma\rangle \otimes |X\rangle$. Operators that act on surface-dependent functionals only affect the surface-dependent part of the wave function and the same with list-dependent ones. Within this representation, the promotion of these operators can be expressed succinctly as follows:

$$\begin{aligned} \hat{E}^{ij}(\vec{x}) &= T^{ij}(\vec{x}, \sigma)\,\mathbb{1}, \\ \hat{A}_{ij}(\vec{x}) &= i\hat{\delta}_{ij}(\vec{x}), \\ \hat{D}(\vec{x}) &= T(\vec{x}, X)\,\mathbb{1}, \\ \partial_i\phi(\vec{x}) &= i\hat{\Delta}_i(\vec{x}). \end{aligned}$$

The canonical variables associated with the Kalb-Ramond field are promoted to surface operators, whereas the ones associated with the Klein-Gordon field, are with list operators. This promotion satisfies the algebra inherited from the Dirac brackets:

$$\begin{aligned} \left[i\delta_{kl}(\vec{y}), T^{ij}(\vec{x},\sigma)\right] &= i\delta^{ij}_{kl}\delta(\vec{x}-\vec{y}) - T^{ij}(\vec{x},\sigma)\delta_{kl}(\vec{y}) = i\delta^{ij}_{kl}\delta(\vec{x}-\vec{y}), \\ \left[i\hat{\Delta}_i(\vec{x}), T(\vec{y}, Y)\right] &= i\delta_{ij}\partial_j\delta^2(\vec{x}-\vec{y}) - T(\vec{y}, Y)\hat{\delta}_i(\vec{x}) = i\delta_{ij}\partial_j\delta^2(\vec{x}-\vec{y}), \end{aligned} \tag{3.2}$$

which follows from the results in (2.13) and (2.24). Note that a dual representation that also satisfies the commutator algebra is possible in which conjugate momenta are promoted to functional derivatives and canonical fields to form factors. The results in the chosen representation can be obtained as well from the dual representation.

As discussed previously, the generator is a topological invariant as its construction remains independent of the metric. The quantized form of $\hat{G}$ is given by the equation:

$$\hat{G} = \int d^2x \, \frac{m}{2} \varepsilon^{ij} \hat{\delta}_{ij}(\vec{x}) \int dz^i \hat{\Delta}_i(\vec{z}) + \int d^2x \, \varepsilon^{ij} \int d\Sigma^{ij}_{\vec{y}} \delta^2(\vec{x} - \vec{y}) \sum_a s_a \delta^2(\vec{y} - \vec{z_a}). \tag{3.3}$$

In this geometrical representation, we expect the generator to present the link between the lists of points and the surfaces. This becomes evident in the second term, which vanishes for signed points outside the boundary of Σ. To elucidate further, this term precisely represents the aggregate of points and anti-points on the surface, delineated as:

$$\int d^2x \, \varepsilon^{ij} \int d\Sigma^{ij}_{\vec{y}} \delta^2(\vec{x} - \vec{y}) \sum_a s_a \delta^2(\vec{y} - \vec{z_a}) = \sum \tilde{s}_a. \tag{3.4}$$

This expression encapsulates the total net contribution from signed points, denoted by $\sum \tilde{s}_a$ that lie within the surface. Zero contribution comes from either no points on the interior region of the surface or from an equal number of points and antipoints inside the boundary. We can think of the points as the boundaries of oriented paths (maybe starting or ending at infinity), in the same sense that loops are considered the boundaries of open 2-surfaces as depicted in 3.1. A nonzero term indicates a net flux (inwards or outwards, depending on the sign) from the interior to the exterior of the boundary.

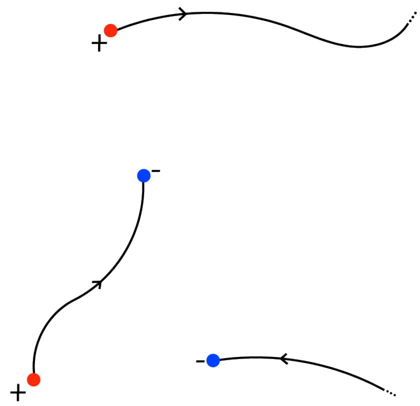

Figure 3.1: Signed points as boundaries of Faraday Lines. Points act as sources, whereas antipoints act as sinks.

Now, as stated above, we can think of oriented paths are Faraday lines that go from points to anti-points. It can be seen that the second term of the generator of duality is a kind of flux as shown in figures 3.2,3.3 where the number is $+1$. Points can be considered sources of flux lines, whereas antipoints can be considered sinks. This represents a projected version of the Gauss law in electrodynamics.

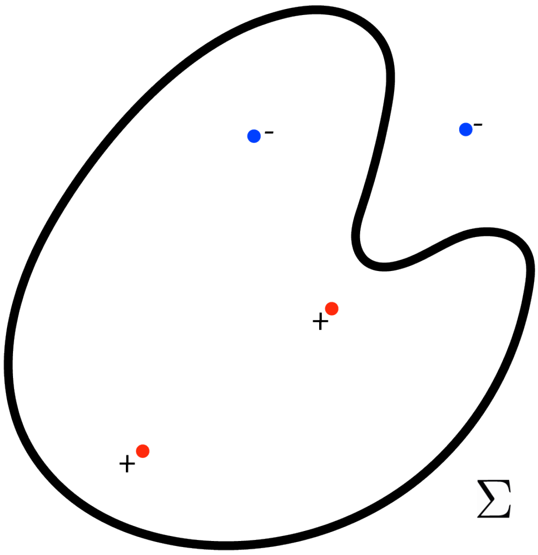

Figure 3.2: Points and antipoints with flux represented by attaching outgoing and incoming lines

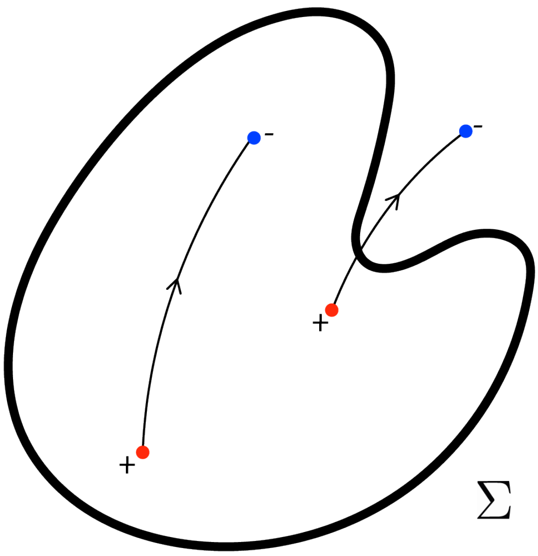

Figure 3.3: Flux from the surface Σ represented by points and antipoints as the boundaries of Faraday Lines

As we can see, the presence of one point and antipoint within the surface gives a net flux of zero, since the lines emanate from the point and fall into the antipoint.

The other boundary of Faraday lines might be in infinity as well since a signed point need not lie outside of the surface.

3.2 Magnetic monopole in the geometric representation

With the tools from 2 we can tackle the framework of 1.4. We shall project our Hilbert space onto path dependent functionals as:

$$\Psi[\gamma] = \langle \gamma | \psi \rangle \,. \tag{3.5}$$

We promote the canonical variables using the prescription of [19]:

$$\begin{aligned} \hat{\Pi}^i &= eT^i(\vec{x}, \gamma)\mathbb{1}, \\ \hat{A}_i &= \frac{1}{e}\delta_i(\vec{x}). \end{aligned} \tag{3.6}$$

We check that this representation satisfies the commutator algebra of (1.78):

$$\begin{aligned} \left[\frac{1}{e}\delta_j(\vec{y}), eT^i(\vec{x}, \gamma)\right]\psi[\gamma] &= \delta^i_j \delta^3(x - y)\psi[\gamma] - T^i(\vec{x}, \gamma)\psi[\gamma \circ u_x]\psi[\gamma] + T^i(\vec{x}, \gamma)\psi[\gamma]\psi[\gamma] \\ &= \delta^i_j \delta^3(\vec{x} - \vec{y})\psi[\gamma] \end{aligned}$$

The constant e is appended to scale the electric field and can be taken to be the fundamental charge, which, in a proper set of units, we can take as one.

The primary constraint of (1.67) is expressed as:

$$\partial_i T^i(\vec{x}, \gamma) = \partial_i \int_\gamma \delta^3(\vec{x} - \vec{y}) dy^i = -\int_\gamma \partial_i^y \delta^3(\vec{x} - \vec{y}) dy^i = 0 \tag{3.7}$$

by the fundamental theorem of calculus, the above integral reduces to:

$$\delta^3(\vec{x} - \vec{y_1}) = \delta^3(\vec{x} - \vec{y_2}) \tag{3.8}$$

where $\vec{y_1}$ and $\vec{y_2}$ are the start and endpoints of the path γ. Thus, the constraint in (1.67) indicates that we shall restrict ourselves to the space of closed loops.

The Hamiltonian in the geometrical representation is:

$$H = \int d^3x \left(\frac{1}{2} (T^i(\vec{x}, \gamma))^2 - \frac{1}{4e^2} (i\Delta_{ij}(\vec{x}) - eb_{ij}(\vec{x}))^2 \right). \tag{3.9}$$

As stated before, for $g = 0$ this reduces to the free Maxwell Electromagnetism. Thus, the effect of the source amounts to replacing the regular loop derivative by a covariant loop derivative taken by:

$$\Delta_{ij} \longrightarrow D_{ij} = \Delta_{ij} - eb_{ij}. \tag{3.10}$$

Equivalently, it is possible to recover the formulation for the free theory by introducing this dependence on the magnetic charge onto the wave function [19]. We define the surface-dependent wave functions as:

$$\begin{aligned}\psi[\Sigma] &= \exp\left(ie\int d\Sigma_y^{ij} b_{ij}\right)\psi[\gamma],\\ &= \exp\left(ieg\int d\Sigma_y^{ij}\epsilon_{ijk}\frac{y^k}{|y|^3}\right)\psi[\gamma],\\ &= \exp\left(\frac{ieg}{4\pi}\Omega(\Sigma)\right)\psi[\gamma].\end{aligned} \tag{3.11}$$

Note that the surface dependence of the wave function is only on the phase factor, we can relate the surface derivative of a surface-dependent functional with the loop derivative of a path-dependent one. We see that:

$$\delta_{ij}(\vec{x})\psi[\Sigma] = \exp\left(\frac{ieg}{4\pi}\right)\left[\frac{ieg}{4\pi}\delta_{ij}\Omega(\Sigma) + \delta_{ij}\right]\psi[\gamma]. \tag{3.12}$$

Using (2.13) we obtain:

$$\begin{aligned}\delta_{ij}\Omega(\Sigma) &= \frac{4\pi}{g}\delta_{ij}\int d\Sigma_{\vec{y}}^{ij} b_{ij}(\vec{y}),\\ &= \frac{4\pi}{g}\int d^3y\, \delta_{ij}(\vec{x})T^{kl}(\vec{y},\Sigma)b_{ij}(\vec{y}),\\ &= \frac{4\pi}{g}\int d^3y\, \frac{1}{2}(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k)b_{ij}(\vec{y}),\\ &= \frac{4\pi}{g}b_{ij}(\vec{x}).\end{aligned} \tag{3.13}$$

Lastly, realize that appending an infinitesimal surface at the point x is equivalent to appending an infinitesimal loop that bounds the surface, independently of the geometry of the surface. Therefore, the surface derivative of the loop-dependent functional can be taken to be its loop derivative. With this, we get that the surface derivative can be expressed as:

$$\delta_{ij}(\vec{x})\psi[\Sigma] = \exp\left(\frac{ieg}{4\pi}\right)(ieb_{ij} + \Delta_{ij})\psi[\gamma], \tag{3.14}$$

which confirms our intuition that the representation in loop space with the covariant derivative D_{ij} is equivalent to the representation in surface space with surface-derivate δ_{ij}.

Furthermore, notice that by going from a loop representation to a surface one, the wave function loses its single-valuedness: a single loop can be the boundary of infinitely many surfaces, so a functional that is uniquely represented in the loop representation is multiply represented in terms of surface dependent functionals by the open surfaces with a common boundary. Take equation (3.11) and consider a surface Σ' that shares the boundary γ with Σ. Since the surface dependency of the wave function lies only on the solid angle subtended by the surface around the monopole in the origin, then the difference in both wave functions is a phase that depends on the difference of solid angles subtended by the surfaces. Alternatively, we can think that the closed surface $\Sigma' \circ (-\Sigma)$. Since the dependence is topological, we can see that the difference :

$$\psi[\Sigma'] = \exp(iegp)\psi(\Sigma), \tag{3.15}$$

where p is the number of times that the closed differential surface $\Delta\Sigma = \Sigma' \circ -\Sigma$ encloses the monopole. This can be expressed through nontrivial boundary conditions: every time the loop goes around a closed trajectory that encloses p times the monopole, the wave function changes by a phase of egp. Thus, we get a multivalued wave function that satisfies the Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi[\Sigma,t] = \int d^3x \left[\frac{1}{2}(T^i(\vec{x},\gamma))^2 - \frac{1}{4e^2}(\Delta_{ij})^2\right]\psi[\Sigma,t], \tag{3.16}$$

with $\psi[\gamma,t]$ obeying:

$$\psi[[S]\gamma] = \exp\{iegp\}\psi[\gamma], \tag{3.17}$$

with $[S]\gamma$ is the closed surface generated by a closed trajectory of the loop γ that wraps p times around the monopole. This is analogous to what occurs in ordinary quantum mechanics for a multiply-connected configuration space, where the wave function is allowed to be multivalued by a phase factor that multiplies the wave function.

Also, note that if Dirac's quantization condition in [2] is satisfied then there is no phase difference for any surface whatsoever since:

$$egp = 2\pi p', \tag{3.18}$$

where $p' \in \mathbb{Z}$. This happens because, in the absence of electric charges, there is no need for a quantization condition such as the one in (3.18). The quantization condition that Dirac derived comes from demanding the electric charge to be single-valued in the presence of the magnetic charge, if there are no electric charges then there is no need for g to be quantized. The same occurs if we consider the dual model with only electric charges [21]. Thus the wave function remains multivalued.

Chapter 4

Conclusions

In summary, this study has delved into the significance and insights that emerge from analyzing quantum field theories with the framework of abelian p-surface spaces and signed points, particularly in the context of the Kalb-Ramond-Klein-Gordon and Schwinger theories. By adopting a geometric perspective, we have not only unearthed valuable information about the models but also set novel avenues for further research.

Through the lens of geometry, we have been able to discern distinct features and characteristics that might have otherwise remained obscured. For instance, in the case of the Kalb-Ramond-Klein-Gordon theory, we have found the explicit form of the generator of duality transformations in 2+1 spacetime dimensions and confirmed it is a topological invariant. We have shown that it corresponds to a link invariant that detects the number of signed points on a surface. Furthermore, by considering signed points as boundaries of Faraday lines, we were able to interpret the invariant as a flux through the surface in a way that bears resemblance to Gauss' Law for electromagnetism. It is noteworthy that this is the first explicit representation of the invariant associated with massive p-form theories.

Similarly, in the Schwinger theory, our geometric analysis has elucidated the representation of magnetic monopoles and their effects as topological defects in the theory. We showed that the presence of a magnetic monopole source in the theory introduces a covariant derivative in the action. Alternatively, we can absorb the effects of the monopole in the wave function, by acting with surface-dependent operators on loop-dependent functionals. This results in recovering the Hamiltonian for the free theory at the cost of the wave function becoming multivalued. The dependence on surfaces is topological, being the solid angle subtended by the surface as seen from the charge, the only important property. Wave functions that depend on different surfaces with the same boundary will differ only by a phase factor that accounts for the number of times the loop wraps around the monopole when describing closed trajectories. This finding resembles the quantum mechanics of multiply-connected configuration spaces, which indicates that the magnetic charges can be seen as topological defects. This model and the one in [21] do not require the charges to be quantized, which preserves multivaluedness. It would be interesting to consider the case with both electric and magnetic sources.

In essence, this work represents an interesting exploration of the mathematical models that describe physical theories. By harnessing the power of geometry, we have managed to get a deeper understanding of different theoretical frameworks. As we move forward, armed with the knowledge gained from this research, we expect to investigate further and dive deeper into different aspects of theoretical physics.

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