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Rompimiento espontáneo de la invarianza de Lorentz y efectos no perturbativos en extensiones del modelo estándard del tipo $\star-$gauge compatibles con los espaciotiempos de Riemann-Cartan

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"My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful." - Hermann Weyl

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Resumen

La teoría cuántica de campos no conmutativos (NCQFT, por sus siglas en inglés) proporciona un marco para explorar los efectos gravitacionales cuánticos mediante la introducción de un parámetro no conmutativo $\theta^{\mu\nu}$ que cuantiza el espaciotiempo. Aunque este enfoque se alinea con la naturaleza cuantizada del espaciotiempo a escalas de Planck, tratar $\theta^{\mu\nu}$ como un tensor de fondo fijo conduce a una violación explícita de la simetría de Lorentz, planteando desafíos teóricos. Motivada por la necesidad de reconciliar la NCQFT con la simetría de Lorentz y los principios de la teoría efectiva de campos, esta tesis propone un mecanismo en el que $\theta^{\mu\nu}$ es modulado dinámicamente por campos escalares regidos por lagrangianos del modelo de sine-Gordon y del modelo lineal sigma. El modelo logra una violación espontánea de la invariancia de Lorentz, asegurando que $\theta^{\mu\nu}$ se anule a bajas energías debido a efectos inducidos por instantones y se active a altas energías. La consistencia del modelo se demuestra a través de su compatibilidad con el mapa de Seiberg-Witten, las Extensiones del Modelo Estándar y la conservación de energía-momentum y espín-momentum angular en el límite de Minowski de los espaciotiempos de Riemann-Cartan.

Palabras clave: Teoría cuántica de campos no conmutativos, violación espontánea de la invariancia de Lorentz, geometría no conmutativa, extensiones del Modelo Estándar, espaciotiempos de Riemann-Cartan.

Abstract

Noncommutative quantum field theory (NCQFT) provides a framework for exploring quantum gravitational effects by introducing a noncommutative parameter $\theta^{\mu\nu}$ that quantizes spacetime. While this approach aligns with the quantized nature of spacetime at the Planck scales, treating $\theta^{\mu\nu}$ as a fixed background tensor leads to explicit Lorentz symmetry violation, raising theoretical challenges. Motivated by the need to reconcile NCQFT with Lorentz symmetry and effective field theory principles, this thesis proposes a novel mechanism where $\theta^{\mu\nu}$ is dynamically modulated by scalar fields governed by sine-Gordon and Linear Sigma Model Lagrangians. The model achieves spontaneous Lorentz invariance violation, ensuring $\theta^{\mu\nu}$ vanishes at low energies through instanton-induced effects and activates at high energies. The consistency of the model is demonstrated through its compatibility with the Seiberg-Witten map, Standard Model Extension, and the conservation of energy-momentum and spin-angular momentum in the Minkowski limit of Riemann-Cartan spacetimes.

Key words: Noncommutative Quantum Field Theory, Spontaneous Lorentz Invariance Violation, Noncommutative Geometry, Standard Model Extensions, Riemann-Cartan spacetimes.

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Introduction

1.1 Motivation and Background

Noncommutative quantum field theory (NCQFT) provides a foundational framework for incorporating quantum properties of spacetime, both algebraically and topologically, making it a powerful candidate for investigating quantum gravity effects near the Planck scale [7, 9, 30]. The motivation for adopting NCQFT stems from the need to reconcile quantum mechanics and general relativity, as both theories predict that spacetime should exhibit a discrete structure due to quantum effects of gravity. At distances on the order of the Planck length, spacetime is expected to deviate from a smooth manifold and acquire a quantized structure—a behavior that NCQFT aims to capture effectively.

At an algebraic level, NCQFT introduces quantization directly through a generalization of the Heisenberg uncertainty principle, extending it to spacetime coordinates via a noncommutative parameter $\theta^{\mu\nu}$. Specifically, NCQFT imposes a noncommutative relation on spacetime coordinates [7, 30]:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1.1}$$

which implies a fundamental limit to the precision with which spacetime intervals can be measured independently. The mathematical foundation of this approach draws on noncommutative geometry, particularly the work of Landi and Connes [7, 24], and utilizes the deformation quantization framework developed by Bayen [3, 22, 28]. In this setting, spacetime observables are treated as operator-valued functions of a C^*-algebra which captures the "fuzzy" geometry at small scales.

Topologically, NCQFT also yields a distinctive structure. The introduction of the noncommutative parameter $\theta^{\mu\nu}$ modifies the topology of spacetime, effectively "smearing" points within a characteristic length scale set by $\theta^{\mu\nu}$. This leads to a nonlocal behavior where points within a certain region cannot be distinguished topologically, aligning with the anticipated lattice-like behavior of spacetime near the Planck scale [7, 21, 24].

Despite its appeal, NCQFT faces significant theoretical challenges that have limited its adoption. One primary difficulty is its apparent incompatibility with Lorentz symmetry in its conventional formulation, which traditionally treats $\theta^{\mu\nu}$ as a fixed background tensor, explicitly breaking Lorentz invariance [4, 23]. This explicit breaking of Lorentz symmetry raises concerns about the physical viability of NCQFT, as it conflicts with key principles of both quantum field theory and relativity. Another major problem is that computing physical observables within a noncommutative algebra of fields is mathematically demanding. Due to this difficulties, Seiberg and Witten postulate a reparametrization capable of turning a noncommutative Lagrangian into a commutative Standard Model Extension (SME). Although the theoretical challenges remain in this reparametrized form, as shown by Carroll [4] and Kostelecký [23], NCQFT can retain theoretical consistency if interpreted within a framework of spontaneous (SLIV) rather than explicit Lorentz invariance violation (ELIV). In the SLIV framework, the noncommutative parameter $\theta^{\mu\nu}$ is promoted to a dynamical field that acquires a non-zero vacuum expectation value, thereby breaking Lorentz symmetry spontaneously by following the dynamic equations describing $\theta^{\mu\nu}$. This approach allows the theory to remain invariant under passive Lorentz transformations while only active Lorentz transformations are affected [4] following the energy scale. Most importantly, by considering a SLIV theory, Kostelecký has shown that all conservation laws are upheld, as well as the CPT theorem. Moreover, it has also been shown that within this framework, one can construct a well-defined gravity sector part of the effective Lagrangian, thus incorporating the quantum effects of gravity [23].

Carroll demonstrated that NCQFTs with an effective SLIV structure can be represented as a subset of SMEs that includes terms for Lorentz violation in the form of background fields [4, 6]. A major goal of this work is to develop a mechanism that enables NCQFT to operate as an SLIV SME theory in line with the insights provided by Carroll and Kostelecký. We propose a model where $\theta^{\mu\nu}$ is modulated dynamically by scalar fields φ^a that acquire energy-dependent vacuum expectation values. These scalar fields are governed by specific dynamics, such as those of the sine-Gordon or Linear Sigma models, which allow for spontaneous symmetry breaking and instanton effects [11, 26, 27]. Moreover, both of these proposed models can be quadratically approximated by a harmonic oscillator; thus, they can be evaluated analytically with minor correction factors [31].

By incorporating instanton solutions, we demonstrate that at low energies, the expectation values $\langle\varphi^a\rangle$ vanish due to instanton-induced symmetry restoration, leading to $\langle\theta^{\mu\nu}\rangle = 0$ and preserving Lorentz invariance. At high energies, where instanton effects are suppressed, $\langle\varphi^a\rangle$ becomes non-zero, resulting in $\langle\theta^{\mu\nu}\rangle \neq 0$ and the emergence of Lorentz-violating effects. This energy-dependent behavior ensures that Lorentz violation is absent at low energies, consistent with experimental constraints, while allowing for possible observable effects at higher energies. Moreover, by dynamically modulating $\theta^{\mu\nu}$, the model achieves spontaneous Lorentz violation without introducing explicit breaking terms or extra gauge symmetries, maintaining compatibility with the Seiberg-Witten map [29] and Kostelecký's framework [23].

This work contributes to the ongoing search for a quantum gravity framework by proposing an NCQFT-based model that integrates spontaneous Lorentz violation and aligns with the structure of Riemann-Cartan spacetimes. By addressing the challenges associated with Lorentz invariance and providing a mechanism for its spontaneous breaking, the model offers a viable avenue for reconciling quantum mechanics and general relativity.

1.2 Structure of the Thesis

This thesis is organized into five chapters, each building upon the previous one to develop a comprehensive understanding of the proposed framework for a SLIV NCQFT.

Chapter 1: Introduction

The introductory chapter is dedicated to providing the motivation and background for the study, with the challenges in reconciling NCQFT with Lorentz invariance due to the fixed noncommutativity parameter $\theta^{\mu\nu}$ being highlighted. The need for a mechanism that allows for energy-dependent noncommutativity is outlined, enabling Lorentz symmetry to be preserved at low energies and spontaneously broken at high energies. The chapter is structured to present the objectives and contributions of the thesis, with emphasis placed on the goal of integrating SLIV into NCQFT in a manner consistent with established theoretical frameworks.

Chapter 2: Theoretical Background

Chapter 2 is focused on reviewing the essential theoretical concepts and mathematical tools necessary for the thesis. An overview of noncommutative geometry and its application in quantum field theory is provided, including the formulation of NCQFT and the role of the noncommutativity parameter

$\theta^{\mu\nu}$. The Weyl-Wigner quantization method and the construction of $\star$-gauge theories are discussed, providing the foundation for understanding how gauge theories are formulated on noncommutative spaces. The Seiberg-Witten map, which relates noncommutative gauge theories to their commutative counterparts by representing noncommutative gauge theories as SMEs, is introduced. The concept of spontaneous Lorentz invariance violation is explored, with its role in SMEs and compatibility with Riemann-Cartan spacetimes being highlighted, such as the conservation of the energy-momentum, spin-density tensor, and CPT theorem. The mathematical tools needed for the computations in Chapter 3 are also introduced, including the semiclassical approximation and instanton solutions, which form the focus of Chapter 3. Limitations such as the Derrick-Hobart theorem are addressed, with a workaround using the ultralocal limit being explored. Finally, the first and second Bogomolny equations are discussed as viable approaches to compute the properties on instantons in the ultralocal limit.

Chapter 3: Modulating the Noncommutativity Parameter

Chapter 3 is dedicated to presenting the proposed mechanism for dynamically modulating the noncommutativity parameter $\theta^{\mu\nu}$. The theoretical framework in which $\theta^{\mu\nu}$ is expressed as a function of scalar fields φ^a, allowing it to acquire energy-dependent behavior, is presented. Two scalar field models, the sine-Gordon model and the Linear Sigma Model, are explored in detail by relating them to results obtained from the quantum harmonic oscillator. The dynamics of these models, including spontaneous symmetry breaking and instanton effects, are analyzed to explain how they lead to the desired modulation of $\theta^{\mu\nu}$. The role of instantons in restoring symmetry at low energies and inducing spontaneous Lorentz violation at high energies is discussed. By examining these models, it is demonstrated how scalar fields can be employed to achieve an energy-dependent $\theta^{\mu\nu}$, addressing the challenges identified in the introduction and producing the desired SLIV NCQFT.

Chapter 4: Consistency of the Model and Its Limitations

Chapter 4 is focused on analyzing the theoretical consistency of the proposed model. The compatibility of the proposed model with the assumptions made, such as the ultralocal limit, the dilute gas approximation, and the cluster decomposition principle, is analyzed. Additionally, the compatibility of the model with the Seiberg-Witten map, despite the modulation of $\theta^{\mu\nu}$, is assessed. The alignment of the model with Kostelecký's framework for spontaneous Lorentz violation, particularly in the context of Riemann-Cartan spacetimes and the conservation of fundamental physical quantities, is discussed. Potential limitations of the model are addressed to ensure its theoretical robustness, while areas requiring further investigation are identified.

Chapter 5: Conclusion

The final chapter is devoted to summarizing the key contributions and findings of the thesis. The significance of dynamically modulating $\theta^{\mu\nu}$ in NCQFT and the manner in which this approach addresses the challenges of explicit Lorentz violation are reflected upon. The implications of integrating spontaneous Lorentz invariance violation into NCQFT are highlighted, with the broader impact of the proposed framework being emphasized.

Chapter 2

Theoretical Background

"In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain." - Hermann Weyl

2.1 Noncommutative Geometry and Quantum Field Theory

2.1.1 Noncommutative Geometry

The development of modern mathematics has enabled a profound abstraction of geometric concepts, allowing for the independent study of space and geometry as fundamental mathematical structures. Classical analytic geometry can be decomposed into these two essential components: the underlying space that serves as a foundation for mathematical discourse, and the framework that enables the construction and analysis of geometric objects within that space. In its most elementary manifestation, this structure is realized through Euclidean space, which provides the mathematical foundation for Euclidean geometry. Within this framework, geometric objects such as lines can be represented through continuous endomorphisms of the space, forming an algebra of functions characterized by specific operational properties.

The intimate relationship between algebraic structures and spatial properties enables the investigation of spatial characteristics through the geometric properties of their associated algebras. For example, the behavior of parallel structures within an algebra can reveal fundamental properties of the underlying space, such as curvature or torsion. This correspondence achieves particular significance in the context of C^*-algebras, where the relationship becomes bijective: the algebraic structure and its corresponding space uniquely determine each other up to isomorphism. This fundamental duality finds its rigorous expression in the Gel'fand–Naimark theorem, which establishes:

Theorem 2.1.1 (Gel'fand–Naimark) *Let A be a C^*-algebra and $\operatorname{Prim} A$ its primitive spectrum. Consider the algebra of continuous, possibly operator-valued, functions $\mathcal{C}(\operatorname{Prim} A)$. Then, A is isometrically *-isomorphic to $\mathcal{C}(\operatorname{Prim} A)$.*

Furthermore, given any topological space M and a primitive spectrum $\operatorname{Prim} \mathcal{C}(M)$, M is homeomorphic to the primitive space $\operatorname{Prim} \mathcal{C}(M)$.

This theorem implies that, given an algebra A, its underlying space can be constructed as $\operatorname{Prim} A$. Similarly, given a topological space M, its overlying algebra can be constructed as $\mathcal{C}(M)$. Hence, space and algebra are equivalent in the sense that one cannot exist without the other, and all properties are uniquely determined by the corresponding pair of space and algebra. In this framework, space, algebra, and geometry are entirely equivalent, with space and algebra as duals, and geometry as their holistic study.

Noncommutative geometry extends this duality by generalizing the concept of algebras. Specifically, as the name suggests, noncommutative geometry involves the holistic study of the space-algebra duality when the C^*-algebra is noncommutative. The Gel'fand–Naimark theorem applies only to commutative algebras, but Landi and Connes successfully generalized this result to noncommutative algebras. While there is no standalone "Noncommutative Gel'fand–Naimark theorem," Landi [24] and Connes [7] dedicate entire books to proving this generalization and exploring its implications. This generalization is neither trivial nor straightforward, yet it preserves the space-algebra duality [7, 24]. Thus, noncommutative geometry refers to the comprehensive study of noncommutative algebras and their corresponding spaces, termed noncommutative spaces.

Although Landi's book extensively covers the properties of noncommutative geometry, our primary interest lies in a specific result: topological indistinguishability. Following Landi's findings, some topological properties of primitive spaces can be described. The crucial property for this work is that if A is a commutative algebra, its primitive space is a Hausdorff space. Conversely, if A is a noncommutative algebra, its primitive space is a Kolmogorov space [24].

Generally, a Hausdorff space is well-suited for describing physical phenomena because any collection of points is topologically distinguishable [21]. In a Hausdorff space, the spatial properties of any object can be precisely determined, such as position and trajectory. By contrast, a Kolmogorov space lacks this property; certain collections of points are not topologically distinguishable [21]. In fact, some collections of distinguishable points cannot be separated and are always contained in the same open sets [21]. This results in regions of space that are fuzzy, causing objects to become delocalized. Such delocalization is desirable in quantum theory, as beyond a certain scale, space itself should become delocalized. Therefore, constructing a quantum field theory over a noncommutative space, aptly termed Noncommutative Quantum Field Theory (NCQFT), aligns with the goals of this work.

To further explore the topological structure of noncommutative spaces, we consider a result by Landi [24]:

Proposition 2.1.2 (Partially Ordered Topological Space) *Let M be at least a Kolmogorov space, equipped with a partial order $\preceq$ over the set. We can then define a basis for a topology as follows:*

$$\Lambda(x) := \{y \in M \mid y \preceq x\}, \quad \forall x \in M.$$

Thus, (M, τ_Λ) forms a topological space known as a partially ordered topological space.

In other words, given any Kolmogorov space we can establish a new topological basis for the space, transforming it into a partially ordered topological space. This is interesting because, a partially ordered topological space can be characterized by two disjoint sets of points: singleton points, which do not precede any other points, and non-singleton points, which precede some other points. Non-singleton points cannot be topologically distinguished from the set of singletons they precede [21, 24]. Consequently, a partially ordered topological space defines a lattice space where singletons form the boundaries of lattice cells, and non-singleton points occupy the interiors of these cells. The lattice structure is more easily visualized by representing the partially ordered topological space as a Hasse diagram.

Definition 2.1.3 (Hasse Diagram for a Partially Ordered Topological Space) *Let (M, τ_Λ) be a partially ordered topological space. For $x, y, z \in M$, the Hasse diagram is constructed by applying the following rules:*

- *If $x \prec y$, then x is at a lower level than y.*
- *If $x \prec y$ and there is no z such that $x \prec z \prec y$, then x is immediately below y, and these points are connected by a link.*

For example, consider the set $\mathbb{Z}$ with the partially-ordered topological basis

$$\Lambda(x_i) = \{x_i\}, \quad \Lambda(y_i) = \{x_i, y_i, x_{i+1}\}, \quad i \in \mathbb{Z}.$$

Its Hasse diagram is represented as follows:

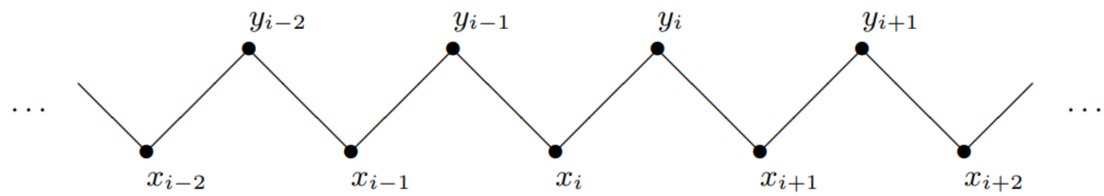

Figure 2.1: Hasse diagram for the set $\mathbb{Z}$

From this diagram, it is evident that no non-singleton y_i can be topologically distinguished from the singletons $\{x_i, x_{i+1}\}$ since we cannot construct an open set that contains y_i without also containing $\{x_i, x_{i+1}\}$. The singletons $\{x_i, x_{i+1}\}$ serve as the boundaries of the lattice cell, while the non-singletons y_i act as the interior points. In this manner, partially ordered topological spaces define a lattice structure.

Proposition 2.1.4 *Let A be a noncommutative C^*-algebra. Then* $\operatorname{Prim} A$ *is homeomorphic to a partially ordered topological space M.*

This proposition, proved in Landi's work [24], reveals the topological structure of noncommutative spaces. They behave similarly to lattice spaces, where each point has a neighborhood such that all points within it are topologically indistinguishable. Therefore, it is more accurate to refer to noncommutative spaces as noncommutative lattices [24].

Thus, we have constructed a noncommutative space or lattice and its associated noncommutative algebra. The next component that we need to construct a quantum field theory is to quantize the algebra [32]. In the next section, we will explore this process of constructing a quantum field theory starting from a noncommutative algebra. The result would be a noncommutative quantum field theory (NCQFT).

2.1.2 Noncommutative Quantum Field Theory

Many different procedures for constructing a NCQFT exist, such as the Weyl-Wigner quantization or the Drinfeld Twist quantization. It is important to note that all of these procedures produce the same NCQFT, although they require different mathematical structures and starting points [1, 22]. Nonetheless, they all fall within Bayen's framework of deformation quantization of Poisson structures [3]. The most notable difference between the Weyl-Wigner and Drinfeld quantizations is that they produce gauge theories in which the group action is applied in different ways, $\star-$gauge and twist-gauge respectively. Nonetheless, it has been shown that these mechanisms of group action are equivalent [10]. In fact, given one of these gauge theories, the other can be constructed within the same framework [10] in what is termed a $\star$-twist gauge.

Weyl-Wigner Quantization

In this work, we focus on the Weyl-Wigner quantization as it is the most straightforward approach and produces gauge theories that are directly compatible with the Seiberg-Witten map, $\star-$gauge theories. The Weyl-Wigner quantization is implemented through a three-step approach. First, we construct a noncommutative algebra that introduces topological indistinguishability into the underlying space. Second, we construct an algebra of fields over $\mathbb{R}^d$. Finally, we find a quantization procedure that maps operators in the noncommutative algebra to fields in the algebra over $\mathbb{R}^d$. This process results in a quantum field theory that is one-to-one equivalent to the noncommutative algebra, thereby forming a NCQFT. This construction is possible because, by imposing the Dirac–von Neumann postulates and a simple set of axioms on C^*-algebras, quantum theories can be constructed within the C^*-algebra formalism [14, 32]. Imposing these conditions allows us to construct a subspace of the C^*-algebra, known as the space of bounded linear operators $\mathcal{BL}$, which forms a Hilbert space [14, 32].

To begin, we construct $\mathbb{R}^d_\theta$, the noncommutative space analogous to Euclidean space $\mathbb{R}^d$.

Definition 2.1.5 **($\mathbb{R}^d_\theta$)** *Let A be a noncommutative C^*-algebra generated by a set of self-adjoint operators $\{x^1, \ldots, x^d\}$ with real spectra. These operators satisfy the commutation relation*

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ is an element of θ, an antisymmetric $d \times d$ matrix with real coefficients. Then, Prim A *is homeomorphic to a noncommutative space denoted $\mathbb{R}^d_\theta$, where the generators of the algebra serve as* *the coordinates of the space. This space is the noncommutative analogue of* $\mathbb{R}^d$.

Furthermore, the dimensions of the noncommutative lattice cells are determined by the coefficients $\theta^{\mu\nu}$,

$$\Delta x^\mu \Delta x^\nu = \ell^{\mu\nu} = |\theta^{\mu\nu}|^{1/2}.$$

Here, θ is known as the noncommutative parameter, as it controls the degree of noncommutativity between the coordinate operators.

Next, we construct a suitable algebra of fields. For Weyl-Wigner quantization, the most appropriate choice is a Schwartz space of fields.

Definition 2.1.6 (Schwartz space) *Let f be a field such that $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$. The Schwartz space $S(\mathbb{R}^n, \mathbb{C})$ is defined as the space of fields f for which, for any multi-indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\mathbb{N}^n$, and a derivation $\boldsymbol{D}$, the seminorm:*

$$\|f\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sup_{\boldsymbol{x}\in\mathbb{R}^n} \left| \boldsymbol{x}^{\boldsymbol{\alpha}} \left(\boldsymbol{D}^{\boldsymbol{\beta}} f \right) (\boldsymbol{x}) \right| < \infty.$$

The Schwartz space is advantageous because, for any field in this space, the Fourier transform is an automorphism. That is, for any field f, it can be identified with its Fourier transform:

$$\tilde{f}(k) = \int \mathrm{d}^D x \, \mathrm{e}^{-\mathrm{i}k_i x^i} f(x). \tag{2.1}$$

This property enables the definition of the Weyl symbol $\hat{\mathscr{W}}[f]$ of a field f, where $\hat{\mathscr{W}}[f]$ is known as Weyl operator of the Wigner field f:

$$\hat{\mathscr{W}}[f] := \int \frac{\mathrm{d}^D k}{(2\pi)^D} \tilde{f}(k) \mathrm{e}^{\mathrm{i}k_i \hat{x}^i}. \tag{2.2}$$

Most notably, the Weyl symbol has the property that the complex exponential field is mapped to the complex exponential operator:

$$\hat{\mathscr{W}} \left[\mathrm{e}^{\mathrm{i}k_i x^i} \right] = \mathrm{e}^{\mathrm{i}k_i \hat{x}^i}. \tag{2.3}$$

This does not precisely mean that complex exponentials are eigenfunctions of the Weyl symbol since the Weyl operator $\mathrm{e}^{\mathrm{i}k_i \hat{x}^i}$, which is operator-valued, inhabits a different space than the Wigner field $\mathrm{e}^{\mathrm{i}k_i x^i}$, which is complex-valued.

Since the Weyl symbol can be thought as an integral transform, we can define a kernel of the transformation, denoted $\hat{\Delta}(x)$, we have:

$$\hat{\Delta}(x) = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \mathrm{e}^{\mathrm{i}k_i \hat{x}^i} \mathrm{e}^{-\mathrm{i}k_i x^i} \to \hat{\mathscr{W}}[f] = \int \mathrm{d}^D x f(x) \hat{\Delta}(x). \tag{2.4}$$

The kernel of this transform can be understood as a mixed basis between operators and fields. Therefore, the Wigner field f is the coordinate space representation of its Weyl operator $\hat{\mathscr{W}}[f]$, the Weyl operator being in the noncommutative algebra $\mathbb{R}^d_\theta$. This means that the Weyl operators conform a representation of the algebra $\mathbb{R}^d_\theta$.

Thereafter, the noncommutativity parameter is introduced once we consider the product of two Weyl operators, this product will contain a term of the form of a product of two kernels $\hat{\Delta}$, which in this case, by the Baker-Campbell-Hausdorff formula is:

$$\hat{\Delta}(x)\hat{\Delta}(y) = \iint \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\mathrm{d}^D k'}{(2\pi)^D} \int \mathrm{d}^D z \mathrm{e}^{\mathrm{i}\left(k+k'\right)_i z^i} \hat{\Delta}(z) \mathrm{e}^{-\mathrm{i}/2\theta^{ij} k_i k'_j} \mathrm{e}^{-\mathrm{i}k_i x^i - \mathrm{i}k'_i y^i} \tag{2.5}$$

or if we assume that θ is invertible:

$$\hat{\Delta}(x)\hat{\Delta}(y) = \frac{1}{\pi^D |\det\theta|} \int \mathrm{d}^D z \hat{\Delta}(z) \mathrm{e}^{-2\mathrm{i}\left(\theta^{-1}\right)_{ij}(x-z)^i(y-z)^j}. \tag{2.6}$$

The next step is to introduce a derivation such that we can define differential operators. The derivation $\hat{\partial}_i$ defined by a set of commutation relations:

$$\left[\hat{\partial}_i, \hat{x}^j\right] = \delta_i^j, \quad \left[\hat{\partial}_i, \hat{\partial}_j\right] = 0, \tag{2.7}$$

so that for a Weyl operator and the kernel we have:

$$\left[\hat{\partial}_i, \hat{\mathscr{W}}[f]\right] = \int \mathrm{d}^D x \partial_i f(x) \hat{\Delta}(x) = \hat{\mathscr{W}}\left[\partial_i f\right], \quad \left[\hat{\partial}_i, \hat{\Delta}(x)\right] = -\partial_i \hat{\Delta}(x). \tag{2.8}$$

By using the definition of the Weyl operators we can now define a trace Tr uniquely by integration over spacetime, this is possible because in a Schwartz space, operators are trace class [8]. That is, the trace of any Weyl operator is finite and independent of basis or representation:

$$\mathrm{Tr}\,\hat{\mathscr{W}}[f] = \int \mathrm{d}^D x f(x), \quad \mathrm{Tr}\,\hat{\Delta}(x) = 1. \tag{2.9}$$

This allow us to define an inverse transformation to the Weyl symbol in terms of the trace:

$$f(x) = \mathrm{Tr}(\hat{\mathscr{W}}[f]\hat{\Delta}(x)), \quad \mathrm{Tr}(\hat{\Delta}(x)\hat{\Delta}(y)) = \delta^D(x-y). \tag{2.10}$$

This inverse is known as the Wigner transformation. Therefore the kernel $\hat{\Delta}$ allows us to construct a one-to-one correspondance between Wigner fields and Weyl operators, given the eponymous Weyl-Wigner correspondance, the core of the Weyl-Wigner quantization.

The final step of this procedure is to unveil the form of the product between two Weyl operators:

$$\mathrm{Tr}\left(\hat{\mathscr{W}}[f]\hat{\mathscr{W}}[g]\hat{\Delta}(x)\right) = \frac{1}{\pi^D|\det\theta|}\iint \mathrm{d}^D y\ \mathrm{d}^D z f(y)g(z)\mathrm{e}^{-2\mathrm{i}\left(\theta^{-1}\right)_{ij}(x-y)^i(x-z)^j}, \tag{2.11}$$

where we can now introduce a product known as the Groenewold-Moyal product $\star$ with the property:

$$\hat{\mathscr{W}}[f]\hat{\mathscr{W}}[g] = \hat{\mathscr{W}}[f\star g]. \tag{2.12}$$

Therefore, we can write the familiar form of the Groenewold-Moyal product in its integral representation:

$$f(x)\star g(x) = \iint \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{d}^D k'}{(2\pi)^D}\tilde{f}(k)\tilde{g}\left(k'-k\right)\mathrm{e}^{-(\mathrm{i}/2)\theta^{ij}k_i k'_j}\mathrm{e}^{\mathrm{i}k'_i x^i}, \tag{2.13}$$

or in differential form [2, 30]:

$$f(x)\star g(x) = f(x)\exp\left(\frac{\mathrm{i}}{2}\overleftarrow{\partial_i}\theta^{ij}\overrightarrow{\partial_j}\right)g(x). \tag{2.14}$$

This gives us some more insight on the form of the non-locality characteristic of noncommutative space. The product of two fields at any given point is influenced by all other points in space as per the integral form of the product $\star$, or all derivatives as per the differential form.

The result of Weyl-Wigner quantization is that we are able to represent the noncommutative algebra $\mathbb{R}^d_\theta$ as a quantum field theory where the products between the fields is replaced by the Groenewold-Moyal product [2, 28, 30]. For example, the Lagrangian for a φ^4 scalar theory could be written as:

$$\mathcal{L}^\star = \frac{1}{2}\partial_\mu\hat{\varphi}\star\partial^\mu\hat{\varphi} - \frac{1}{2}m^2\hat{\varphi}\star\hat{\varphi} - \frac{\lambda}{4!}\hat{\varphi}\star\hat{\varphi}\star\hat{\varphi}\star\hat{\varphi}, \tag{2.15}$$

where we have used hatted fields $\hat{\varphi}$ to remind the reader that these fields are commutative fields embedded into noncommutative space $\mathbb{R}^d_\theta$ by using the star product $\star$.

Finally, although this construction assumes that θ is constant, it can be generalized for any form of θ by considering deformation quantization of an appropriate Poisson structure by following the Kontsevich procedure [1, 22, 28, 30]. This result is paramount for this work, as the model that we will propose will amount as allowing θ to be dynamic over spacetime.

$\star$-Gauge Theory

The extension of NCQFT to gauge theories initially appears straightforward. Starting with a traditional gauge theory, such as a Yang-Mills theory, one replaces all products in the Lagrangian between matter and gauge fields with the noncommutative $\star$-product, as we have done for the $\hat{\varphi}^4$ scalar Lagrangian. For instance, in the case of Quantum Electrodynamics (QED), the Lagrangian becomes:

$$\mathcal{L}^\star_{\text{QED}} = \frac{1}{2} i\bar{\hat{\psi}} \star \gamma^\mu D^\star_\mu \hat{\psi} - m\bar{\hat{\psi}} \star \hat{\psi} - \frac{1}{4q^2}\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}. \tag{2.16}$$

However, this straightforward extension encounters a significant issue. The theory is no longer invariant under the gauge group $U(1)$. Under a gauge transformation U, the matter fields transform as:

$$\bar{\hat{\psi}} \star \hat{\psi} \to \left(\bar{\hat{\psi}} U^\dagger\right) \star \left(U\hat{\psi}\right) \neq \bar{\hat{\psi}} \star \hat{\psi}. \tag{2.17}$$

To address this problem, the action of the gauge group must be modified in terms of the $\star$-product, to produce a $\star-$gauge. The gauge transformation for $\hat{\psi}$ is redefined as:

$$\delta^\star_\omega \hat{\psi}(x) := i\omega^a(x) T^a \star \hat{\psi}(x). \tag{2.18}$$

The gauge transformation U is then expressed as:

$$U_\star(x) := \sum_{n=0}^{\infty} \frac{i^n}{n!} \omega^{a_1}(x) \star \cdots \star \omega^{a_n}(x) T_{a_1} \cdots T_{a_n} := e_\star^{i\omega^a(x)T^a}, \quad U_\star \star U_\star^\dagger = U_\star^\dagger \star U_\star = 1. \tag{2.19}$$

This ensures that gauge symmetry is restored:

$$\bar{\hat{\psi}} \star \hat{\psi} \to \bar{\hat{\psi}} \star U_\star^\dagger \star U_\star \hat{\psi} = \bar{\hat{\psi}} \star \hat{\psi}. \tag{2.20}$$

Moreover, as it is clear from the definition of the Groenewold-Moyal product, computing any physical observable in terms of products of fields is mathematically challenging. Thus, the Seiberg-Witten map allows for the reparametrization of the NCQFT into a SME, such that physical observables can be computed using standard techniques. This is a major advantage of $\star-$gauge theories, as they can be directly mapped via the conjectured relationship.

Seiberg-Witten Map

The Seiberg-Witten map posits that the noncommutative fields $\hat{A}$ and $\hat{\psi}$ can be expressed in terms of their commutative counterparts A and ψ [1, 18] by identifying gauge orbits. Hence, this relationship is unique modulo the gauge orbits of the theory. This relationship is determined by solving the following highly non-trivial differential equation:

$$\hat{\psi}\left(A+\delta_{\omega}A, \psi+\delta_{\omega}\psi\right)=\hat{\psi}(A,\psi)+\delta_{\omega}^{\star}\hat{\psi}(A,\psi), \tag{2.21}$$

by performing small variations in term of $\theta^{\mu\nu}$ results in the following differential equation in terms of the matter field ψ, gauge connection A and the covariant derivative D_{ν} [1]:

$$\delta\theta^{\mu\nu}\frac{\partial\hat{\psi}}{\partial\theta^{\mu\nu}}=-\frac{1}{4}\delta\theta^{\mu\nu}\hat{A}_{\mu}\star\left(\partial_{\nu}\hat{\psi}+D_{\nu}\hat{\psi}\right). \tag{2.22}$$

It is important to remark that given this differential equation, it could be thought that introducing a dynamic $\theta^{\mu\nu}$ would produce a different reparametrization of $\hat{A}$ and $\hat{\psi}$ than when $\theta^{\mu\nu}$ is constant. This is not the case, since the reparametrization works in the same way as long as both theories have the same gauge orbits, they will be mapped to the same gauge theory [1]. Thus, we have an important constraint in the form of $\theta^{\mu\nu}$, whatever the modulation is proposed, it should no introduce extra gauge symmetries.

Explicit solutions are generally unknown for most gauge theories. However, for simple gauge groups, such as $U(1)$, which describes QED, the solution can be obtained perturbatively in terms of $\theta^{\mu\nu}$:

$$\begin{cases}\hat{A}_{\mu}=A_{\mu}-\dfrac{1}{2}\theta^{\alpha\beta}A_{\alpha}\left(\partial_{\beta}A_{\mu}+F_{\beta\mu}\right)+\mathcal{O}(\theta^{2}),\\ \hat{\psi}=\psi-\dfrac{1}{2}\theta^{\alpha\beta}A_{\alpha}\partial_{\beta}\psi+\mathcal{O}(\theta^{2}).\end{cases} \tag{2.23}$$

This reparametrization casts the theory into a form consistent with the Standard Model Extension (SME), where noncommutative effects appear as additional terms in the Lagrangian. Thus, for noncommutative QED (NCQED), it can be expressed, via the Seiberg-Witten map as [4, 15]:

$$\begin{aligned}\mathcal{L}^{\star}_{\text{QED}} \to \mathcal{L}_{\text{QED, SW}} =& \frac{1}{2} i\bar{\psi}\gamma^{\mu}D_{\mu}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &- \frac{1}{8} iq\theta^{\alpha\beta}F_{\alpha\beta}\bar{\psi}\gamma^{\mu}D_{\mu}\psi + \frac{1}{4} iq\theta^{\alpha\beta}F_{\alpha\mu}\bar{\psi}\gamma^{\mu}D_{\beta}\psi \\ &+ \frac{1}{4} mq\theta^{\alpha\beta}F_{\alpha\beta}\bar{\psi}\psi \\ &- \frac{1}{2} q\theta^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu} + \frac{1}{8} q\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\theta^2) \\ &= \mathcal{L}_{\text{QED}}(\psi, A) + \mathcal{L}_{\text{LIV}}(\theta, \psi, A) := \mathcal{L}_{\text{QED, SME}},\end{aligned} \tag{2.24}$$

which is precisely the definition of a SME which are characterized by being composed of a Standard Model Lagrangian and a Lorentz Invariance Violating sector [6, 23].

2.2 Standard Model Extensions, Riemann-Cartan Spacetimes and Conservation Laws

What is interesting about the Seiberg-Witten map is that the resulting effective Lagrangian describes a SME as defined by Kostelecký [4, 6, 23]. An SME is an effective Lagrangian that is composed of at least two different sectors. The first sector is a Standard Model Lagrangian, while the second sector contains all possible Lorentz-violating terms constructed in terms of the fields of the first sector. This second sector is modulated by a tensor, which in our case for a NCQFT is simply $\theta^{\mu\nu}$, called the coefficient for Lorentz violation:

$$\mathcal{L}_{\text{SME}}(\psi, A) := \mathcal{L}_{\text{SM}}(\psi, A) + \mathcal{L}_{\text{LIV}}(\theta, \psi, A). \tag{2.25}$$

By reinterpreting Lorentz invariance under the framework of Riemann-Cartan theories in the vierbein formalism, these Seiberg-Witten-type effective Lagrangians that derive from NCQFT theories can be shown to remain invariant under passive Lorentz transformations, while not under active Lorentz transformations [4]. This result stems directly from the inclusion of the noncommutative parameter θ in the Lorentz-violating sector of the Lagrangian. Then, it can be shown that θ transforms correctly as a 2-form under passive transformations but transforms as a constant under active transformations, hence introducing Lorentz Invariance Violation (LIV) [4, 23].

An important result by Kostelecký and Carroll is that if this Lorentz violation is spontaneous

(SLIV), rather than explicit (ELIV), then the quantum theory described by the Lagrangian is compatible with Riemann-Cartan spacetimes [4, 23]. Compatibility means that the theory upholds the CPT theorem, as well as all conservation laws and currents. Moreover, it can be shown that in a SLIV theory, we are able to construct the Bianchi identities and, from there, construct an action for gravity such as the Hilbert-Einstein action or the Palatini action [23]. In fact, it is expected that an SME must contain a pure gravity sector [23]. Thus, in general we can write a SME Lagrangian as being composed of two sectors, matter and gravity:

$$\begin{cases} \mathcal{L}_{\text{SME}}(\psi, A, \theta, e^a_\mu, \omega^{ab}_\mu) = \mathcal{L}_{\text{matter}}(\psi, A, \theta) + \mathcal{L}_{\text{gravity}}(e^a_\mu, \omega^{ab}_\mu), \\ \mathcal{L}_{\text{matter}}(\psi, A, \theta) = \mathcal{L}_{\text{SM}}(\psi, A) + \mathcal{L}_{\text{LIV}}(\theta, \psi, A), \end{cases} \tag{2.26}$$

where e^a_μ is the vierbein and ω^{ab}_μ is the spin conection in Riemann-Cartan spacetimes.

The problem here is that even though Carroll and Kostelecký conjecture that a mechanism to turn NCQFT into a SLIV theory could be constructed, they do not introduce any candidates for such a mechanism. The main objective of this work is to introduce a mechanism to turn NCQFT into a SLIV theory compatible with Kostelecký's results.

2.2.1 Riemann-Cartan Theory and Conservation Laws

The significance of Riemann-Cartan theory, as a generalization of General Relativity, is that it is constructed using the vierbein formalism. Hence, it allows incorporating a spinorial representation into the usual group of symmetries of General Relativity $GL(4, \mathbb{R})$ needed to incorporate matter fields, as well as allowing for a mathematical distinction between passive and active transformations [23].

In the vierbein formalism, gravitational fields are taken to be the vierbein e^a_μ and their spin connection ω^{ab}_μ, where Latin indices are local reference frames and Greek indices are global reference frames. Then, the corresponding Riemann-Cartan spacetimes are described in terms of both a curvature tensor $R^\kappa_{\lambda\mu\nu}$ and a torsion tensor $T^\lambda_{\mu\nu}$. The usual General Relativity spacetimes, such as the Minkowski spacetime, can be obtained by setting the torsion and curvature tensors to 0. This more general framework allows introducing dynamical torsion and curvature as propagation of the vierbein and spin connection [23]. Another interesting aspect is that, in this formalism, active transformations behave like local gauge transformations. Therefore, Lorentz invariance violation under active transformations is equivalent to local gauge invariance violation [23].

The study of conservation laws in the presence of spontaneous Lorentz invariance violation (SLIV) requires understanding the behavior of the energy-momentum tensor and related conservation laws when Lorentz symmetry is broken through vacuum expectation values of tensor fields. Here, active transformations affect the localized fields, while passive transformations change the reference frame. The following presentation outlines these derivations and emphasizes the conclusion that under SLIV in active transformations, the dynamics of the Lorentz-breaking tensor ensure the conservation of energy-momentum and spin-density tensors [6, 23].

For the moment, we consider a 1-tensor k_x responsible for the SLIV for simplicity. It is important to remember that our model requires a 2-tensor $\theta^{\mu\nu}$ but as we will see, the following results are independent of the rank of the tensor responsible for the SLIV.

In this case, the Lorentz-violating action S_{LIV} component of the matter sector S_{matter} can be expressed as the passive-covariant integral of the coefficient k_x for Lorentz violation with an operator J^x, that is:

$$S_{\mathrm{LIV}} \supset \int d^4x\, e\, k_x J^x, \tag{2.27}$$

where J^x is the associated operator, written in terms of the vierbein, spin connection, and SM fields. Moreover, e stands for the determinant of the vierbein field. For NCQFT, this term would simply be $e\theta_{xy}J^{xy}$, as seen for example in the Lagrangian $\mathcal{L}_{\mathrm{QED,\ SME}}$. This form guarantees that the theory remains invariant under passive Lorentz transformations [23]. Following this construction, the Lorentz-violating terms in the matter action can be expressed as:

$$S_{\mathrm{LIV}} = \int d^4x\, e\, k_x J^x(f^y, e^\mu_a D_\mu f^y), \tag{2.28}$$

where J^x involves the matter fields f_y and their covariant derivatives $e^\mu_a D_\mu$ in the vierbein formalism. This structure ensures consistency with the vierbein and spin connection to maintain general coordinate invariance [23]. When all fields and background structures, including the coefficients for Lorentz violation, are varied, the variation of the matter action becomes:

$$\delta S_{\mathrm{LIV}} = \int d^4x e \left({T_e}^{\mu\nu} e_{\nu a} \delta e^a_\mu + \frac{1}{2} {S_\omega}^\mu{}_{ab} \delta {\omega_\mu}^{ab} + e J^x \delta k_x \right), \tag{2.29}$$

where ${T_e}^{\mu\nu}$ is the energy-momentum tensor, and ${S_\omega}^\mu{}_{ab}$ is the spin-density tensor in Riemann-Cartan spacetimes [23]. We now have two different cases that could induce such varition, infinitesimal active or passive Lorentz transformations.

If we consider the previous variation to be induced by an infinitesimal active Lorentz transformation parametrized by ϵ^{ab}, then the relevant infinitesimal changes in the vierbein, spin connection, and coefficients for Lorentz violation take the form:

$$\delta e_{\mu}{}^{a} = -\epsilon^{a}{}_{b} e_{\mu}{}^{b}, \quad \delta\omega_{\mu}{}^{ab} = -\epsilon^{a}{}_{c}\omega_{\mu}{}^{cb} + \epsilon^{cb}\omega_{\mu}{}^{a}{}_{c} + \partial_{\mu}\epsilon^{ab}, \quad \delta k_x = -\frac{1}{2}\epsilon^{ab}\left(X_{[ab]}\right)^{y}{}_{x} k_y. \tag{2.30}$$

By substituting these variations into the expression for $\delta S_{\mathrm{matter}}$ and simplifying, we obtain the symmetry condition for the energy-momentum tensor:

$$T_{e}{}^{\mu\nu} - T_{e}{}^{\nu\mu} = \left(D_{\alpha} - T^{\beta}_{\beta\alpha}\right) S_{\omega}{}^{\alpha\mu\nu} + e^{\mu a}e^{\nu b} k_x \left(X_{[ab]}\right)^{x}{}_{y} J^{y}. \tag{2.31}$$

This indicates that the antisymmetric part of the energy-momentum tensor depends on the spin-density tensor and the coefficients for Lorentz violation [23]. In the limit of flat Minkowski spacetime, this equation simplifies to:

$$\Theta_{c}{}^{\mu\nu} - \Theta_{c}{}^{\nu\mu} = \partial_{\alpha} S_{c}{}^{\alpha\mu\nu} + k_x \left(X^{[\mu\nu]}\right)^{x}{}_{y} J^{y}, \tag{2.32}$$

where $\Theta_{c}^{\mu\nu}$ is the canonical energy-momentum tensor, and $S_{c}^{\alpha\mu\nu}$ is the canonical spin-density tensor [23]. This equation corresponds to the conservation of spin-angular momentum.

On the other hand, if the variation was induced by an infinitesimal passive Lorentz transformation parametrized by ϵ^{μ}, then the variations are given by Lie derivatives:

$$\begin{gathered} \delta e_{\mu}{}^{a} = \mathcal{L}_{\epsilon} e_{\mu}{}^{a} = e_{\nu}{}^{a}\partial_{\mu}\epsilon^{\nu} + \partial_{\nu} e_{\mu}{}^{a}\epsilon^{\nu}, \quad \delta\omega_{\mu}{}^{ab} = \mathcal{L}_{\epsilon}\omega_{\mu}{}^{ab} = \omega_{\nu}{}^{ab}\partial_{\mu}\epsilon^{\nu} + \partial_{\nu}\omega_{\mu}{}^{ab}\epsilon^{\nu}, \\ \delta k_x = \mathcal{L}_{\epsilon} k_x = \epsilon^{\mu}\partial_{\mu} k_x. \end{gathered} \tag{2.33}$$

Substituting these into the variation of the action and simplifying results in the covariant conservation law:

$$\left(D_{\mu} - T^{\lambda}{}_{\lambda\mu}\right) T_{e}{}^{\mu}{}_{\nu} + T^{\lambda}{}_{\mu\nu} T_{e}{}^{\mu}{}_{\lambda} + \frac{1}{2} R^{ab}{}_{\mu\nu} S_{\omega}{}^{\mu}_{ab} - J^{x} D_{\nu} k_x = 0. \tag{2.34}$$

In the case of a Minkowski spacetime, where curvature and torsion are absent, this equation simplifies to:

$$\partial_{\mu}\Theta_{c}{}^{\mu\nu} = J^{x}\partial^{\nu} k_x, \tag{2.35}$$

which corresponds to the conservation of energy-momentum. Both of these conservation laws derived here are compatible and consistent in the Minkowski spacetime limit [23]. Furthermore, a discussion of their consistency and compatibility with general Riemann-Cartan spacetimes can be found in [23].

Now, in an SLIV theory, the spontaneous Lorentz invariance violation leaves the associated conserved currents unaffected. Thus, both conservation laws remain conserved, absent of terms depending on k^x [23]. This can be seen directly from the fact that the variation of S_{LIV} is absent of terms containing δf^x, since k_x in the case of SLIV must follow the equations of motion, therefore the action must also lack terms deriving from δk_x [23]. That is, in the case of SLIV theory, the conservation laws are reduced to

$$\Theta_c{}^{\mu\nu} - \Theta_c{}^{\nu\mu} = \partial_\alpha S_c{}^{\alpha\mu\nu},\ \partial_\mu \Theta_c{}^{\mu\nu} = 0, \tag{2.36}$$

since this result is independent on the rank of the tensor responsible for the SLIV, it remains valid in the case for SMEs deriving from NCQFT.

The results presented in these derivations indicate that the conservation of energy-momentum and spin-angular momentum are preserved in the presence of SLIV. This conservation is ensured as long as the coefficients for Lorentz violation correspond to vacuum expectation values that satisfy their dynamical equations of motion [23]. Furthermore, the ability to construct the Einstein-Hilbert action within the SME framework underscores that gravitational dynamics remain consistent with the modified Bianchi identities, enabling the formulation of gravitational field equations that incorporate Lorentz-violating effects. This comprehensive compatibility of conservation laws and geometric consistency is what Kostelecký means when he states that theories with SLIV are compatible with Riemann-Cartan spacetimes. The SME thus confirms that conserved quantities involving both energy-momentum and spin-angular momentum are maintained under SLIV, as long as the dynamics of the Lorentz-violating tensors adhere to their governing equations.

The final symmetry to be preserved by following Kostelecký's framework is the CPT symmetry. This symmetry can only be understood in the Minkowski limit of Riemann-Cartan spacetimes, as its generalization to curved spacetimes is still unknown [23]. In the Minkowski limit, it is a well-known fact that CPT violation is tied directly to the form of the coefficients for Lorentz violation. In our case, $\theta^{\mu\nu}$ has only two Lorentz indices and is therefore classified as CPT-even under Kostelecký's framework. This means that the CPT theorem is upheld even in the presence of SLIV [6, 23].

2.3 Semiclassical Approximation in Quantum Field Theory and Instantons

The semiclassical approximation is a fundamental tool in quantum field theory that bridges the gap between classical field theories and their quantum counterparts. It allows for the exploration of quantum phenomena by considering quantum fluctuations around classical solutions of the field equations. This approximation is particularly powerful when dealing with non-perturbative effects that cannot be captured by standard perturbation theory, such as tunneling processes mediated by instantons. In this section, we present the theoretical framework of the semiclassical approximation, focusing on its properties and the role of finite action solutions in dominating the path integral.

2.3.1 Semiclassical Approximation

The starting point of the semiclassical approximation is the path integral formulation of quantum field theory, where the generating functional $Z[J]$ is expressed as an integral over all possible field configurations:

$$Z[J] = \int \mathcal{D}\varphi \, \exp\left\{ i \left[S[\varphi] + \int d^4x \, J(x)\varphi(x) \right] \right\}, \tag{2.37}$$

where $S[\varphi]$ is the action functional of the field $\varphi(x)$, and $J(x)$ is an external source. Then, the vacuum-to-vacuum transition amplitude without sources is given by:

$$Z[0] = \int \mathcal{D}\varphi \, e^{iS[\varphi]} = \lim_{t\to\infty} \langle 0, t | 0, -t \rangle. \tag{2.38}$$

In the semiclassical approximation, we expand the action around classical solutions $\varphi_{\rm cl}(x)$ that extremize the action, satisfying the classical equations of motion:

$$\left. \frac{\delta S[\varphi]}{\delta \varphi(x)} \right|_{\varphi=\varphi_{\rm cl}} = 0. \tag{2.39}$$

We then write the field as $\varphi(x) = \varphi_{\rm cl}(x) + \eta(x)$, where $\eta(x)$ represents the quantum fluctuations around the classical solution. Substituting this expansion into the action and expanding to second order in η, we obtain:

$$S[\varphi] = S[\varphi_{\rm cl}] + \int d^4x \, \left. \frac{\delta S}{\delta \varphi(x)} \right|_{\varphi_{\rm cl}} \eta(x) + \frac{1}{2} \int d^4x \, d^4y \, \left. \frac{\delta^2 S}{\delta \varphi(x) \delta \varphi(y)} \right|_{\varphi_{\rm cl}} \eta(x)\eta(y) + O(\eta^3). \tag{2.40}$$

The linear term in η vanishes due to the classical equations of motion, simplifying the expansion. The quadratic term represents the leading contribution of quantum fluctuations, and higher-order terms are neglected in the semiclassical approximation. The path integral then becomes:

$$Z[0] \approx e^{iS[\varphi_{\rm cl}]} \int \mathcal{D}\eta \, \exp\left\{ \frac{i}{2} \int d^4x \, d^4y \; \frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}\bigg|_{\varphi_{\rm cl}} \eta(x)\eta(y) \right\}. \tag{2.41}$$

This integral is Gaussian and can be formally evaluated, leading to:

$$Z[0] \approx e^{iS[\varphi_{\rm cl}]} \left[\det \left(\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}\bigg|_{\varphi_{\rm cl}} \right) \right]^{-\frac{1}{2}}. \tag{2.42}$$

The determinant arises from integrating over the quantum fluctuations and represents the quantum corrections to the classical action, this is precisely the form in which we will compute the tunneling amplitude of instantons in this work.

The semiclassical approximation is based on several key properties. Firstly, in the limit where $\hbar \to 0$, the exponential $e^{iS[\varphi]/\hbar}$ becomes rapidly oscillating unless $S[\varphi]$ is stationary. Therefore, the path integral is dominated by field configurations near the classical solutions $\varphi_{\rm cl}$ that extremize the action. Secondly, the approximation assumes that quantum fluctuations $\eta(x)$ around the classical solution are small and can be treated perturbatively. This is valid when the action $S[\varphi_{\rm cl}]$ is large compared to $\hbar$, ensuring that the exponential factor does not suppress the contributions from classical paths. Thirdly, the classical solutions considered must have finite action to contribute significantly to the path integral. Finite action ensures that the exponential factor $e^{iS[\varphi_{\rm cl}]/\hbar}$ does not vanish or become infinite, making the contribution from $\varphi_{\rm cl}$ meaningful.

Finite action solutions, such as instantons, play a crucial role in the semiclassical approximation because they dominate the path integral in certain regimes. These solutions are non-perturbative and cannot be accessed through standard perturbation theory around the trivial vacuum. The importance of finite action solutions arises from the exponential weighting in the path integral:

$$Z[0] \simeq e^{iS[\varphi_{\rm cl}]/\hbar} \; (1 + O(\hbar)) \,. \tag{2.43}$$

When $S[\varphi_{\rm cl}]$ is large compared to $\hbar$, the phase oscillations in the exponent can lead to constructive or destructive interference, depending on the path. However, paths near the classical solution contribute coherently, reinforcing their dominance in the path integral. Finite action solutions often correspond to tunneling processes between different vacua or topological sectors of the theory.

2.3.2 Instantons

Instantons are non-trivial finite action solutions to the Euclidean equations of motion that are localized in Euclidean spacetime. They possess several important properties. They represent tunneling events that connect different vacua of the theory and are inherently non-perturbative, as they cannot be expanded in powers of the coupling constant. Their key property relies on the fact that their Euclidean action is finite:

$$S_E[\varphi_{\text{inst}}] = \int d^4x_E \left[\frac{1}{2}(\partial_\mu\varphi_{\text{inst}}\ \partial^\mu\varphi_{\text{inst}}) + V(\varphi_{\text{inst}})\right] < \infty, \tag{2.44}$$

where we use the Euclidean metric $\partial_\mu\partial^\mu = \delta^{\mu\nu}\partial_\mu\partial_\nu$ and the Wick rotation $\tau = -it$. Instantons often carry a topological charge or winding number that classifies the solution, an integer-valued quantity that remains invariant under continuous deformations of the field configuration [5, 25, 27]. Although this topological property is instrumental in our understanding of instantons, it is only important in gauge theories. Therefore, we have chosen to omit it here, as the proposed models in this work involve only non-gauge scalar theories.

Another important aspect of instantons is that they represent well-localized configurations in Euclidean space. That is, they manifest as sharp perturbations localized in a comparatively small region. This perturbations are stable and persistent. This is the motivation behind calling them pseudo-particles [25].

Schema for Instantons

Incorporating instanton effects into quantum field theories involves several key steps and can be done by following this schema:

1. Identifying Instanton Solutions

Firstly, we need to identify instanton solutions by solving the Euclidean equations of motion for the field $\varphi(x)$. The equation to be solved is:

$$\left.\frac{\delta S_E[\varphi]}{\delta\varphi(x)}\right|_{\varphi=\varphi_{\text{inst}}} = 0, \tag{2.45}$$

where $S_E[\varphi]$ is the Euclidean action obtained by performing a Wick rotation ($t \to -i\tau$) on the Minkowski action. This equation represents the stationary points of the Euclidean action, and its solutions $\varphi_{\text{inst}}(x)$ are the instanton configurations. These solutions must satisfy appropriate boundary conditions, typically approaching different minima of the potential at Euclidean time infinities to ensure finite action:

$$\lim_{x\to\pm\infty} \varphi_{\text{inst}}(x) = \varphi_{\pm}, \tag{2.46}$$

where $\varphi_{\pm}$ are the vacuum states of the theory.

2. Calculating the Classical Action

Secondly, we calculate the classical action by evaluating the Euclidean action for the instanton solution:

$$S_E[\varphi_{\text{inst}}] = \int d^4x \left[\frac{1}{2}(\partial_\mu \varphi_{\text{inst}}\ \partial^\mu \varphi_{\text{inst}}) + V(\varphi_{\text{inst}})\right]. \tag{2.47}$$

This integral computes the total action associated with the instanton configuration over Euclidean spacetime. Calculating the classical action is essential because it quantifies the leading-order contribution of the instanton to the path integral. Moreover, the action corresponds directly to the energy of the configuration in Euclidean spacetime.

3. Computing the Tunneling Amplitude and Fluctuation Determinant

Thirdly, we compute the tunneling amplitude associated with the instanton transition between vacua:

$$\langle\varphi_+|e^{-\tau\mathcal{H}/\hbar}|\varphi_-\rangle = N\int D\varphi e^{-S_E[\varphi_{\text{inst}}]/\hbar}, \tag{2.48}$$

where the functional measure $[D\varphi]$ represents functional integration with respect to all functions φ that satisfy the boundary conditions for instantons. Moreover, N is a normalization constant. Now, we can consider φ to be composed of φ_{inst} plus some fluctuation η such that

$$\varphi(x) = \varphi_{\text{inst}} + \sum_n c_n \eta_n(x), \tag{2.49}$$

where η represents eigenfunctions of the second variation $\delta^2 S_E[\varphi_{\text{inst}}]$. The tunneling amplitude can then be expressed as [5, 26, 31]:

$$\langle\varphi_+|e^{-\tau\mathcal{H}/\hbar}|\varphi_-\rangle = Ne^{-S_E[\varphi_{\text{inst}}]/\hbar}\left[\det\left(-\partial_\mu\partial^\mu + V''(\varphi_{\text{inst}})\right)\right]^{-\frac{1}{2}}, \tag{2.50}$$

where the determinant accounts for quantum fluctuations around the instanton configuration.

4. Considering Multi-instanton Contributions and Applying the Dilute Gas Approxi-

mation

In the dilute gas approximation, we assume that instantons are rare, widely separated, and non-interacting. This approximation allows us to treat the instantons as a "gas" of non-overlapping, isolated events. This introduces a correction factor such that the tunneling amplitude becomes proportional to:

$$\int_{a_1 < \cdots < a_n} da_1 \cdots da_n = \frac{A^n}{n!}, \tag{2.51}$$

where A represents the collective coordinate of the instantons, such as position. The dilute gas approximation is crucial for our work because, under this approximation, the correlation functions are significantly affected by the instanton contributions [11].

2.3.3 The Derrick-Hobart Theorem

The Derrick-Hobart theorem is a fundamental result in classical field theory that places stringent constraints on the existence of stable, localized, finite-energy solutions—like instantons—in scalar field theories. Specifically, the theorem states that in spatial dimensions greater than or equal to two, nontrivial static solutions of finite energy cannot exist for a wide class of scalar field theories with canonical kinetic terms. This theorem has significant implications for the construction of models that rely on such solutions, necessitating alternative approaches to circumvent its restrictions.

Consider a real scalar field $\varphi(\mathbf{x})$ described by the energy functional:

$$E[\varphi] = \int d^D x \left[\frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi + V(\varphi) \right]. \tag{2.52}$$

We seek static solutions $\varphi(\mathbf{x})$ that minimize the energy, satisfying the Euler-Lagrange equation:

$$\frac{\delta E}{\delta \varphi(x)} = -\partial^\mu \partial_\mu \varphi + \frac{dV}{d\varphi} = 0. \tag{2.53}$$

To understand the Derrick-Hobart theorem, we consider the scaling properties of the energy functional. Assume that $\varphi(x)$ is a finite-energy solution described by an energy functional with non-negative kinetic and potential terms. We perform a scale transformation by introducing a parameter $\lambda > 0$ and defining a scaled field:

$$\varphi_\lambda(x) = \varphi(\lambda x). \tag{2.54}$$

The energy of the scaled field is:

$$E[\varphi_\lambda] = \int d^D x \left[\frac{1}{2} \partial_\mu \varphi_\lambda \, \partial^\mu \varphi_\lambda + V(\varphi_\lambda) \right]. \tag{2.55}$$

We compute the kinetic and potential energies separately under this scaling. For the kinetic term we have:

$$\begin{aligned} E_{\text{kin}}[\varphi_\lambda] &= \frac{1}{2} \int d^D x \, \partial_\mu \varphi_\lambda \, \partial^\mu \varphi_\lambda \\ &= \frac{1}{2} \lambda^2 \int d^D x \, \partial_\mu \varphi(\lambda x) \, \partial^\mu \varphi(\lambda x) \\ &= \frac{1}{2} \lambda^2 \lambda^{-D} \int d^D y \, \partial_\mu \varphi(y) \, \partial^\mu \varphi(y) \\ &= \lambda^{2-D} E_{\text{kin}}[\varphi], \end{aligned} \tag{2.56}$$

where we have changed variables to $y = \lambda x$ and used $d^D x = \lambda^{-D} d^D y$. Similarly, the potential energy scales as:

$$\begin{aligned} E_{\text{pot}}[\varphi_\lambda] &= \int d^D x \, V(\varphi_\lambda(x)) \\ &= \int d^D x \, V(\varphi(\lambda x)) \\ &= \lambda^{-D} \int d^D y \, V(\varphi(y)) \\ &= \lambda^{-D} E_{\text{pot}}[\varphi]. \end{aligned} \tag{2.57}$$

Therefore, the total energy of the scaled field configuration is:

$$E[\varphi_\lambda] = E_{\text{kin}}[\varphi_\lambda] + E_{\text{pot}}[\varphi_\lambda] = \lambda^{2-D} E_{\text{kin}}[\varphi] + \lambda^{-D} E_{\text{pot}}[\varphi] = E(\lambda). \tag{2.58}$$

We see that the energy is a function of the scaling parameter λ, and its behavior is determined by the dimensionality D of the system. To determine if $\varphi(x)$ is a stationary point under scaling, we consider the derivative of the energy with respect to λ at $\lambda = 1$:

$$\begin{aligned} \left. \frac{dE(\lambda)}{d\lambda} \right|_{\lambda=1} &= \frac{d}{d\lambda} \left(\lambda^{2-D} E_{\text{kin}}[\varphi] + \lambda^{-D} E_{\text{pot}}[\varphi] \right) \Big|_{\lambda=1} \\ &= (2-D)\lambda^{1-D} E_{\text{kin}}[\varphi] - n\lambda^{-D-1} E_{\text{pot}}[\varphi] \Big|_{\lambda=1} \\ &= (2-D) E_{\text{kin}}[\varphi] - D E_{\text{pot}}[\varphi]. \end{aligned} \tag{2.59}$$

For $\varphi(\mathbf{x})$ to be a stationary point under scaling, this derivative must vanish:

$$(2-D)E_{\rm kin}[\varphi] - DE_{\rm pot}[\varphi] = 0. \tag{2.60}$$

This is known as Derrick's scaling condition. We need to analyze the behavior of this scaling condition considering three different dimensionalities $D = 1, 2$ or $D \geq 3$

For $D = 1$, the condition becomes:

$$(2-1)E_{\rm kin}[\varphi] - 1E_{\rm pot}[\varphi] = 0 \implies E_{\rm kin}[\varphi] - E_{\rm pot}[\varphi] = 0. \tag{2.61}$$

Since both $E_{\rm kin}[\varphi] \geq 0$ and $E_{\rm pot}[\varphi] \geq 0$, this condition can be satisfied by nontrivial solutions where $E_{\rm kin}[\varphi] = E_{\rm pot}[\varphi]$. In one spatial dimension, nontrivial finite-energy static solutions can exist because the energy functional can attain a minimum.

For the marginal case $D = 2$, the scaling condition simplifies to:

$$(2-2)E_{\rm kin}[\varphi] - 2E_{\rm pot}[\varphi] = 0 \implies -2E_{\rm pot}[\varphi] = 0 \implies E_{\rm pot}[\varphi] = 0. \tag{2.62}$$

This implies that the potential energy must vanish:

$$E_{\rm pot}[\varphi] = \int d^D x\, V(\varphi(x)) = 0. \tag{2.63}$$

If $V(\varphi) \geq 0$, this can only occur if $V(\varphi(x)) = 0$ almost everywhere in the integration measure. Therefore, the field must reside in the minima of the potential almost everywhere, leading to a trivial solution. Consequently, nontrivial finite-energy static solutions do not exist in $D = 2$ unless the potential permits $V(\varphi) = 0$ for nontrivial $\varphi(x)$.

For $n \geq 3$, the coefficient $(2 - D)$ is negative. The scaling condition becomes:

$$DE_{\rm pot}[\varphi] = (2-D)E_{\rm kin}[\varphi]. \tag{2.64}$$

Since $(2 - D) < 0$ for $D \geq 3$, the right-hand side is negative. However, $E_{\rm pot}[\varphi] \geq 0$, which implies that $E_{\rm kin}[\varphi] \leq 0$, contradicting the fact that $E_{\rm kin}[\varphi] \geq 0$. Therefore, the scaling condition cannot be satisfied unless both $E_{\rm kin}[\varphi] = E_{\rm pot}[\varphi] = 0$, leading to a trivial solution like for the marginal case.

Therefore, the Derrick-Hobart theorem demonstrates that in scalar field theories with canonical kinetic terms, nontrivial, finite-energy, static solutions are unstable and cannot exist in spacetime dimensions $D \geq 2$. This result is significant because it constrains the types of finite-energy solutions that can be supported in such theories. To construct stable, localized solutions in higher dimensions, one must consider theories with modified kinetic terms, include gauge fields, or explore other mechanisms to circumvent the limitations imposed by the theorem.

One viable approach is to modify the theory or consider specific limits where the assumptions of the theorem no longer apply. A particularly effective method is the ultralocal limit, where spatial derivatives are neglected, effectively reducing the dimensionality of the theory to $0+1$, thus allowing for the existence of instantons as per the theorem. In the next section, we will explore the ultralocal limit in detail and demonstrate how it enables the construction of scalar field theories that incorporate instanton effects while overcoming the limitations set by the Derrick-Hobart theorem.

2.4 The Ultralocal Limit

The concept of the ultralocal limit in quantum field theory, developed by Klauder, provides an alternative framework to the traditional treatment of scalar field theories by focusing on configurations where spatial fluctuations are suppressed. This approach is crucial for addressing the challenges posed by the Derrick-Hobart theorem, which prohibits the existence of nontrivial finite-energy solutions in scalar field theories for dimensions $n \geq 2$. In the ultralocal limit, the Lagrangian is modified such that only temporal fluctuations remain significant, effectively reducing the field theory to a $0+1$-dimensional system where instanton solutions can manifest as they do in quantum mechanics. This section explores the theoretical underpinnings of the ultralocal limit and its implications, drawing heavily on the work by Klauder, Francisco and Gamboa [12, 13, 19, 20].

To introduce the ultralocal limit, consider the standard kinetic term of a scalar field theory:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi). \tag{2.65}$$

We can define the characteristic length of spatial and temporal fluctuations as:

$$\frac{1}{T} \sim \left|\frac{\partial_t \varphi}{\varphi}\right|, \quad \frac{1}{L} \sim \left|\frac{\nabla \varphi}{\varphi}\right|. \tag{2.66}$$

Next, we introduce an infrared cutoff ℓ that modulates the spatial sector by proposing the following effective Lagrangian [13, 19, 20]:

$$\mathcal{L}_{\text{eff, kin}} = \frac{1}{2}(\partial_t \varphi)^2 + \sum_{n=0}^{\infty} c_n \left(\frac{\ell}{L}\right)^{2n} (\nabla\varphi)^{2n}. \tag{2.67}$$

Choosing an appropriate ℓ such that $\frac{\ell}{L} \ll 1$, the effective Lagrangian can be simplified to exclude spatial fluctuations within a region of size ℓ^3:

$$\mathcal{L}_{\text{ul, kin}} = \frac{1}{2}(\partial_t \varphi)^2. \tag{2.68}$$

This is known as the ultralocal approximation or limit. This limit allows for the existence of instantons in scalar field theories by reducing the dimensionality of the theory, thus avoiding the constraints of the Derrick-Hobart theorem. The applicability of this limit to our model will be explicitly examined in a later section, where the characteristic fluctuation times and lengths will be fine-tuned.

Smearing and Operator-Valued Distributions

The principal characteristic of the ultralocal limit is the vanishing of spatial fluctuations. That is, fields located in space evolve independently at each point in space [19, 20]. Consider a field operator $\varphi(x,t)$ defined over a space of arbitrary dimensionality. In ultralocal models, the Hamiltonian $\mathcal{H}$ is defined without terms involving spatial derivatives, such as $\nabla\varphi(x,t)$. The ultralocal property of this Hamiltonian implies that the dynamics of the field at each point x are independent of the dynamics at any other point. At a fixed spatial point $\overline{x}$, the field behaves as a purely time-dependent field $\varphi_{\overline{x}}(t)$.

To reconstruct a well-behaved spacetime field, we use spatial smearing with respect to a real test function $f(x)$ with compact support $\overline{x} \in \Omega \subset \mathbb{R}^n$. By convolution, the ultralocal fields $\varphi_{\overline{x}}(t)$ within the compact support Ω are transformed into a spacetime field that exhibits spatial correlation within Ω, an operator-valued distribution [16]:

$$\varphi(\chi, t) = \int_{\Omega} \varphi_{\overline{x}}(t) f(\overline{x})\, d\overline{x}. \tag{2.69}$$

This test function serves to average out local fluctuations and make $\varphi(\chi,t)$, with $\chi \in \Omega$ serving as a spatial coordinate, well-behaved in the context of quantum theory as an operator-valued distribution [16, 19, 20]. The function $f(x)$ ensures that while the field $\varphi_{\overline{x}}(t)$ remains independent at different spatial points, the smeared field $\varphi(\chi,t)$ can still exhibit spatial homogeneity in observables within Ω.

This homogeneity does not imply interaction between points; rather, it reflects a uniform response across space when considering smeared observables [16, 19, 20]. Since Ω is compact, this homogeneity is valid only within a finite region and at a specific time, preserving causality. True independence is retained for observables with disjoint supports for their smearing functions [12, 19, 20].

The physical significance of this construction is that in the ultralocal limit, the $n+1$-dimensional field theory reduces to a collection of independent $0+1$-dimensional quantum systems [13, 19, 20]. Each point x behaves as if governed by its own time evolution without spatial influence. The test function $f(x)$ integrates over these independent systems, creating a smeared field that, when observed, appears homogeneous across space [19, 20] and gives an analogous representation to $\varphi(x,t)$.

This approach implies that non-perturbative effects in such theories, including phenomena like instantons, can be analyzed by studying their counterparts in simpler quantum mechanical systems. For this work, we are not interested in the explicit form of the smearing function or the smeared field $\varphi(\chi,t)$ but rather in the result that, in the ultralocal limit, the dynamics of fields can be studied as a reduced dimensionality problem in quantum mechanics, accounting for the loss of degrees of freedom by assuming spatial homogeneity within a compact subspace.

2.4.1 Instantons in the Ultralocal Limit

Following the discussion of the previous section, we will now compute the general form of instanton solutions in a scalar field theory in the ultralocal limit. To this end, consider the standard form of the Euclidean scalar field Lagrangian in the ultralocal limit:

$$\mathcal{L}_{\text{ul}} = \frac{1}{2}\left(\partial_\tau \varphi\right)^2 + V(\varphi). \tag{2.70}$$

Then, its action would be

$$S_{\text{ul}} = \int d\tau \mathcal{L}_{\text{ul}} = \int d\tau \left(\frac{1}{2}(\partial_\tau \varphi)^2 + V(\varphi)\right). \tag{2.71}$$

By performing the first variation over the action for this Lagrangian, we can find the generic equation of motion for a scalar field as:

$$\delta S_{\text{ul}} = 0 \implies \partial_\tau^2 \varphi - \partial_\varphi V = 0. \tag{2.72}$$

This is precisely the form of the classical equations of motion of a particle moving over the potential $-V$ by making the identification $\varphi \leftrightarrow x$. Moreover, by making this identification, we see that the action transforms into the Euclidean action for instanton solutions in a quantum mechanical system:

$$S'_{\text{ul}} = \int d\tau L'_{\text{ul}} = \int d\tau \left(\frac{1}{2}(\partial_\tau x)^2 + V(x)\right). \tag{2.73}$$

In this way, we see that the reduced dimensionality of the ultralocal limit of our scalar field theory is equivalent to a quantum mechanical system. Therefore, instantons in the ultralocal limit have the same form as its quantum mechanical analogues with the identification $x \leftrightarrow \varphi$.

By applying Noether's theorem on the τ-translation invariance of the system, we find that the total energy must be conserved:

$$\frac{1}{2}(\partial_\tau \varphi)^2 - V(\varphi) = 0 \implies \partial_\tau \varphi_{\text{inst}} = \pm\sqrt{2V(\varphi_{\text{inst}})} \tag{2.74}$$

This equation is known as the first Bogomolny equation and it is the differential equation that characterizes instanton solutions [25]. This is an integrable equation whose solution is then:

$$\int \frac{d\varphi}{\sqrt{2V(\varphi)}} = \pm \int d\tau. \tag{2.75}$$

Moreover, this equation also allows us to compute its action, or equivalently in Euclidean spacetime, its energy, in what is called the second Bogomolny equation:

$$S[\varphi_{\text{inst}}] = 2\int d\tau\ V(\varphi_{\text{inst}}) = \int d\tau\ (\partial_\tau \varphi_{\text{inst}})^2 = E[\varphi_{\text{inst}}]. \tag{2.76}$$

In this work, we will use this equation to derive the general form of the instanton solutions for the proposed models in the ultralocal limit. This produces time-dependent instantons, their spacetime counterparts are obtained by performing the appropriate smearing to create an operator-valued distribution [16, 19, 20] in such a way that their spatial behavior is homogeneous. In this work, we will not perform the smearing explicitly, but we will use this fact to check for the consistency of the model proposed.

Chapter 3

Model

3.1 Modulation of Noncommutative Parameter via Scalar Fields

The proposed model, termed Modulated Noncommutative Quantum Field Theory (MNCQFT), introduces a dynamic mechanism for the noncommutativity parameter $\theta^{\mu\nu}$. In NCQFT, $\theta^{\mu\nu}$ is treated as a fixed background parameter. However, MNCQFT allows $\theta^{\mu\nu}$ to evolve dynamically through modulation by scalar fields φ^a residing in a moduli space. This modulation is mathematically expressed as:

$$\theta^{\mu\nu}(x) = \sum_a \varphi^a(x)\theta^a, \tag{3.1}$$

where θ^a are elements from a basis of antisymmetric matrices, forming the linear space Θ to which $\theta^{\mu\nu}$ belongs. This framework allows $\theta^{\mu\nu}$ to vary according to energy scales, controlled by the behavior of the scalar fields φ^a. Moreover, it is important to recall that although we suggest that this fields exist outside of the Minkowski spacetime of the SM, they modify the behavior of the theory directly since they would appear in all terms of the SLIV sector of the SME where the matter fields are required to be contracted with $\theta^{\mu\nu}$.

The objective of allowing $\theta^{\mu\nu}$ to be dynamic is to recover the following expected behavior incorporating non-perturbative effects via the semiclassical approximation, the expectation value of the scalar fields behaves as follows:

$$\langle\varphi^a\rangle \begin{cases} \langle\varphi^a\rangle_{E<\Lambda} = 0 \implies \langle\theta^{\mu\nu}\rangle = 0, \\ \langle\varphi^a\rangle_{E>\Lambda} \neq 0 \implies \langle\theta^{\mu\nu}\rangle \neq 0. \end{cases} \tag{3.2}$$

This behavior implies that at low energies, the expectation value $\langle\varphi^a\rangle$ vanishes, leading to a zero physical expectation value for $\theta^{\mu\nu}$ and thus suppressing all Lorentz-violating terms in the SME Lagrangian. As a result, the theory effectively reduces to the Standard Model at low energy scales. Above a critical energy threshold Λ, however, $\langle\varphi^a\rangle$ becomes non-zero, yielding a non-zero $\langle\theta^{\mu\nu}\rangle$ and the emergence of Lorentz-violating interactions. This mechanism introduces SLIV into NCQFT, aligning with the conditions for compatibility with Riemann-Cartan theory [4, 23].

3.1.1 Scalar Field Models

The decision to modulate $\theta^{\mu\nu}$ using scalar fields is motivated by both theoretical considerations and phenomenological viability. Allowing $\theta^{\mu\nu}$ to vary independently would necessitate constructing a pure Yang-Mills-type theory with a complex gauge structure, lacking phenomenological support. By employing scalar fields, the model avoids introducing new symmetries beyond those present in the Standard Model, simplifying the theoretical framework while retaining the capacity to explore noncommutative effects.

Moreover, allowing $\theta^{\mu\nu}$ to vary independently would imply the addition of a gauge symmetry to the Standard Model (SM). This is undesirable because, under the Seiberg-Witten map, the theory would necessarily map to a different SME with different gauge orbits. This would jeopardize the main objective of this work, as it relies on the SME analyzed by Carroll, which is compatible with Kostelecký's framework.

Finally, under the ultralocal limit, instantons in scalar field theories are phenomenologically identical to instantons in non-relativistic quantum mechanics. Therefore, their behavior can be interpreted intuitively, allowing for the construction of symmetry arguments, which are paramount to achieving the desired behavior of the model. Additionally, this approach simplifies calculations. Instantons in non-relativistic quantum mechanics are well-understood mathematically, enabling verification of most calculations with the literature, particularly references such as [5, 11, 25, 27]. This ensures the model's alignment with current scientific understanding. If generalizations are needed, this work provides a robust foundation due to its simplicity and the strong scientific consensus surrounding instantons in non-relativistic quantum mechanics.

3.2 Harmonic Oscillator and Fluctuation Determinant

In this section, we will present the instanton solutions for a quantum harmonic oscillator. This analysis is essential because, as we will see, most computations can be extended by considering a simple phenomenological argument.

As explained before, instantons behave as well-localized phenomena. They transition between different vacuum states within a very short timespan and remain most of their time well-localized in the vacuum states. Therefore, it is a well-founded assumption to analyze their properties by considering how they behave within a single vacuum state and introducing a correction factor to account for their tunneling. That is, we reduce the problem to studying the harmonic oscillator coupled to a correction factor. Both models we propose can be approximated by the harmonic oscillator at their vacuum states. Most notably, this approach allows us to compute the fluctuation determinant required for instantons, after which we consider a correction factor to extend these results to the other proposed models.

Let us consider the following Euclidean Lagrangian:

$$\begin{cases} \mathcal{L} = \dfrac{1}{2}\left(\partial_\tau \varphi\right)^2 + V(\varphi), \\ V(\varphi) = \dfrac{1}{2}\mu^2\varphi^2. \end{cases} \tag{3.3}$$

We propose the following expansion around events between an initial time $-T/2$ and final time $T/2$, with $T \to \infty$, as described in the schema for instantons:

$$\varphi(\tau) = \varphi_{\text{cl}} + \sum_n c_n \eta_n(\tau), \tag{3.4}$$

where we have substituted φ_{inst} with φ_{cl} because, for a single vacuum state, we do not have instantons. Instead, we consider the classical path from the semiclassical approximation. By considering the fluctuations to be eigenfunctions of the second variation of the action, we have:

$$\langle \varphi_f | \, e^{-\tau \mathcal{H}/\hbar} \, | \varphi_i \rangle = N e^{-S_E[\varphi_{\text{cl}}]/\hbar} \left[\det\left(-\partial_\tau^2 + V''\left(\varphi_{\text{cl}}\right)\right)\right]^{-\frac{1}{2}}. \tag{3.5}$$

By choosing eigenfunctions of the second variation, we can now write the fluctuation determinant as:

$$\begin{aligned} N\left[\det\left(-\partial_\tau^2 + V''\left(\varphi_{\rm cl}\right)\right)\right]^{-1/2} &= N\prod_n \lambda_n^{-1/2} \\ &= N\prod_n \left(\frac{\pi^2 n^2}{T^2\hbar^2}\right)^{-1/2} \prod_n \left(1+\frac{V''(\varphi_{\rm cl})^2 T^2}{\pi^2 n^2}\right)^{1/2}, \end{aligned} \tag{3.6}$$

where $\lambda_n = \frac{\pi^2 n^2}{T^2\hbar^2} + V''(\varphi_{\rm cl})^2$ are the eigenvalues associated with the fluctuations η_n. Using the fact that [5, 31]:

$$N\prod_n \left(\frac{\pi^2 n^2}{T^2\hbar^2}\right)^{-1/2} = \int \frac{dp}{2\pi} e^{-p^2 T\hbar/2} = \frac{1}{\sqrt{2\pi T\hbar}}, \tag{3.7}$$

and the hyperbolic sine identity:

$$\sinh(\pi y) = \pi y \prod_n \left(1+\frac{y^2}{n^2}\right), \tag{3.8}$$

we can write:

$$\begin{aligned} N\big[\det\left(-\partial_\tau^2 + V''\left(\varphi_{\rm cl}\right)\right)\big]^{-1/2} &= \frac{1}{\sqrt{2\pi T\hbar}}\left(\frac{\sinh(V''\left(\varphi_{\rm cl}\right)T)}{V''\left(\varphi_{\rm cl}\right)T}\right)^{-1/2} \\ &= \sqrt{\frac{V''\left(\varphi_{\rm cl}\right)}{\pi\hbar}}(2\sinh(V''\left(\varphi_{\rm cl}\right)T))^{-1/2} \\ &= \sqrt{\frac{V''\left(\varphi_{\rm cl}\right)}{\pi\hbar}} e^{-V''(\varphi_{\rm cl})T/2}\left(1+\frac{1}{2}e^{-2V''(\varphi_{\rm cl})T}+\cdots\right). \end{aligned} \tag{3.9}$$

Then as $T\to\infty$, at first order we recover:

$$N\left[\det\left(-\partial_\tau^2 + V''\left(x_{\rm cl}\right)\right)\right]^{-1/2} = \left(\frac{V''(\varphi_{\rm cl})}{\pi\hbar}\right)^{1/2} e^{-V''(\varphi_{\rm cl})T/2} = \Delta N \tag{3.10}$$

As stated before, due to the phenomenological behavior of instantons, if for a general theory with degenerate vacuum states $N\geq 2$, if it can be approximated as a harmonic oscillator near these vacua, then we can assume that Δ will have the same form and will be corrected by a factor K[31]:

$$\Delta N \to K\Delta N, \tag{3.11}$$

such that the tunneling amplitude is also corrected as:

$$\langle\varphi_f|\, e^{-\tau\mathcal{H}/\hbar}\,|\varphi_i\rangle = K\Delta N e^{-S_E[\varphi_{\rm cl}|/\hbar]}. \tag{3.12}$$

Moreover, if given the case that we can consider n multi-instanton effects, we will also assume that:

$$\Delta N \to K^n \Delta N \tag{3.13}$$

In this work, we have chosen two models with the required asymptotic behavior near their vacuum states: a sine-Gordon model and a Linear Sigma Model $O(N)$, ensuring the validity of these assumptions.

3.3 Sine-Gordon Model

The first model that we propose is a model based on the sine-Gordon Lagrangian. In this model, we propose that each scalar field φ^a is described by their own independent sine-Gordon Lagrangian without mixed interaction terms. Then, the full Lagrangian would be just a summation over the independent sectors. For this model, we will find that the vanishing of the expectation value of the scalar fields φ^a is a result from the acquired symmetry of the topological vacuum $|\Omega\rangle$ whose construction is mediated by the effects of instantons. To this aim, consider the Lagrangian for a particular φ^a given by:

$$\begin{cases} \mathcal{L} = \dfrac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - V(\varphi), \\ V(\varphi) = \dfrac{\mu^2}{\omega^2}\left(1 - \cos \omega\varphi\right), \end{cases} \tag{3.14}$$

where μ is a coupling constant setting the scale of the potential, and ω determines the periodicity. Then, in the ultralocal limit its Euclidean Lagrangian becomes:

$$\mathcal{L}_{\text{ul}} = \frac{1}{2}\left(\partial_\tau \varphi\right)^2 + \frac{\mu^2}{\omega^2}\left(1 - \cos \omega\varphi\right). \tag{3.15}$$

Moreover, the sine-Gordon model requires φ to be an angular field:

$$\varphi(\tau) : \mathbb{R} \to S^1. \tag{3.16}$$

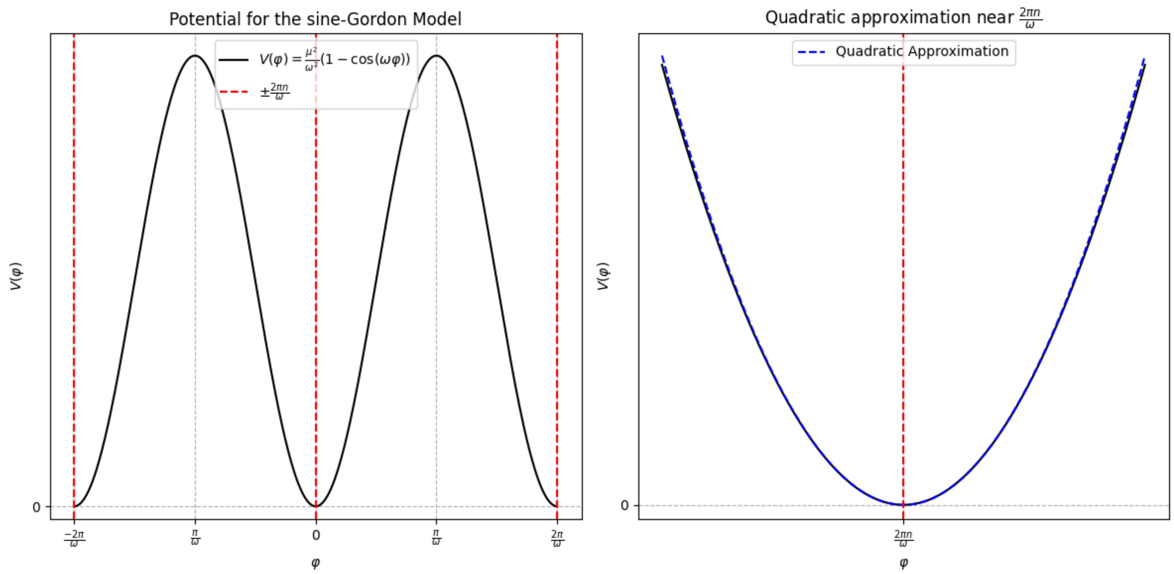

Figure 3.1: Potential for the sine-Gordon Model extended in a line with its quadratic approximation near the minima.

In this case, by imposing the boundary conditions corresponding to tunneling between the degenerate vacua:

$$\begin{cases} \lim_{\tau \to -\infty} \varphi = 0 = \varphi_-, \\ \lim_{\tau \to +\infty} \varphi = \dfrac{2\pi}{\omega} = \varphi_+. \end{cases} \tag{3.17}$$

we can use the first Bogomolny equation to find the instanton solution

$$\begin{aligned} \pm \int d\tau &= \int \frac{d\varphi}{\sqrt{\frac{2\mu^2}{\omega^2}\left(1 - \cos\omega\varphi\right)}} \\ &= \frac{1}{\mu} \int \frac{1}{t} \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1}{2\mu} \int \left(\frac{1}{s+1} - \frac{1}{s-1} \right) \\ \implies \pm\Delta\tau &= \frac{1}{2\mu} \ln \left| \frac{1-s}{1+s} \right| \Bigg|_s , \end{aligned} \tag{3.18}$$

which implies that the instanton solution takes the form:

$$\frac{1 - \cos\frac{\omega\overline{\varphi}}{2}}{1 + \cos\frac{\omega\overline{\varphi}}{2}} = e^{\pm\mu(\tau - \tau')} \implies \overline{\varphi}(\tau) = \frac{4}{\omega} \arctan\left(e^{\pm\mu(\tau-\tau')} \right), \tag{3.19}$$

where τ' is the time around which the instanton is centered, corresponding to the moduli space of instantons in this model. Moreover, in this case, the characteristic time of the instanton is $\tau_0 = \frac{1}{\mu}$.

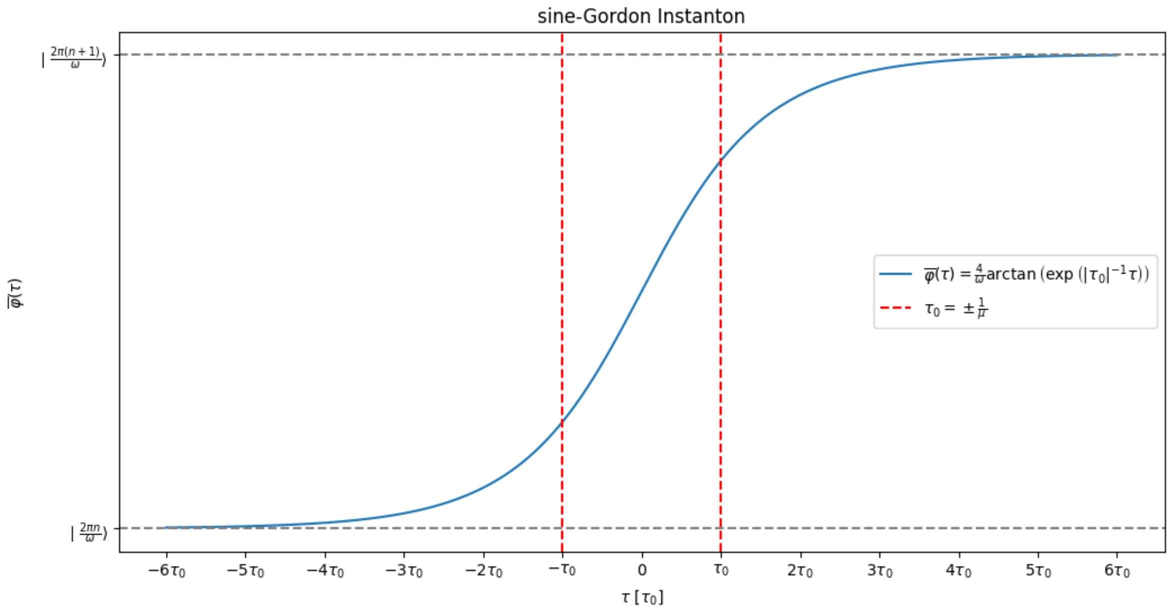

Figure 3.2: Single instanton for the sine-Gordon model.

The action for this instanton is finite and corresponds directly to its energy. Using the action side of the second Bogomolny equation:

$$
\begin{aligned}
S\left[\overline{\varphi}\right] &= \int_{-\infty}^{\infty} d\tau \left[\frac{1}{2}\left(\partial_\tau \overline{\varphi}\right)^2 + \frac{\mu^2}{\omega^2}\left(1 - \cos\omega\overline{\varphi}\right)\right] \\
&= 2\int_{-\infty}^{\infty} d\tau \ \frac{\mu^2}{\omega^2}(1 - \cos\omega\overline{\varphi}) \\
&= \frac{16\mu}{\omega^2} = E[\overline{\varphi}].
\end{aligned}
\tag{3.20}
$$

This finite action implies significant contributions from instantons in the semiclassical limit. The contribution taking the form of tunneling between the degenerate vacua of the theory, with amplitude:

$$
\langle\varphi_+|\, e^{-\tau\mathcal{H}/\hbar}\, |\varphi_-\rangle = K\Delta N e^{-\frac{16\mu}{\omega^2\hbar}} = \xi, \tag{3.21}
$$

where K is the correction term resulting from approximating the vacuum states of the sine-Gordon model to the harmonic oscillator. Moreover, due to the symmetry of the model, we will have that the tunneling amplitude between any two contiguous vacuum states $|\frac{2\pi n}{\omega}\rangle \to |\frac{2\pi(n+1)}{\omega}\rangle$ will be ξ.

3.3.1 Construction of the Topological Vacuum and Vanishing Expectation Value

In order to obtain the desired behavior of a vanishing expectation value, we will consider the tunneling effects of the instantons. These tunneling effects can be incorporated into the model by constructing an effective Hamiltonian describing tunneling:

$$\mathcal{H}_{\text{eff}} = E_0 \sum_N |N\rangle\langle N| - \xi \sum_N \left(|N+1\rangle\langle N| + |N\rangle\langle N+1|\right), \tag{3.22}$$

where $|N\rangle$ represents the different vacuum states admitted by the potential. The periodic potential allows for a superposition state following Bloch's theorem [27]:

$$|\Omega\rangle = \sum_N A e^{iN\Omega}|N\rangle, \tag{3.23}$$

where A is a normalization constant unimportant to this analysis. Moreover, this Bloch state is an eigenstate of $\mathcal{H}_{\text{eff}}$:

$$\begin{aligned}
\mathcal{H}_{\text{eff}}|\Omega\rangle &= \left[E_0 \sum_N |N\rangle\langle N| - \xi \sum_N \left(|N+1\rangle\langle N| + |N\rangle\langle N+1|\right)\right] \sum_M A e^{iM\Omega}|M\rangle \\
&= E_0 \sum_{N,M} |N\rangle\langle N|A e^{iM\Omega}|M\rangle - \xi \sum_{N,M} \left(|N+1\rangle\langle N|A e^{iM\Omega}|M\rangle + |N\rangle\langle N+1|A e^{iM\Omega}|M\rangle\right) \\
&= A E_0 \sum_N e^{iN\Omega}|N\rangle - A\xi \left(e^{i\Omega} + e^{-i\Omega}\right) \sum_N e^{iN\Omega}|N\rangle \\
&= A E_0|\Omega\rangle - 2A\xi \cos\Omega|\Omega\rangle.
\end{aligned} \tag{3.24}$$

The energy expectation value for this state is minimized when $\Omega = 0$, yielding the topological vacuum:

$$|\Omega = 0\rangle = A \sum_N |N\rangle, \tag{3.25}$$

which represents the symmetrical superposition of all degenerate vacua $|N\rangle$.

From the construction of this topological vacuum state $|\Omega\rangle$ it is straightforward to see that it is invariant under the dihedral group $\text{Dih}(\omega) = g$. Let's consider first the rotational generator for the group $T_{\text{rot}}(g)$, whose action induces a rotation of $\frac{2\pi}{\omega}$. If we consider the vacuum states $|N\rangle$, then this rotation shifts each state $N \to N+1$:

$$T_{\mathrm{rot}}(g)|N\rangle = |N+1\rangle \implies T_{\mathrm{rot}}(g)|\Omega\rangle = A\sum_{N}|N+1\rangle = |\Omega\rangle. \tag{3.26}$$

On the other hand, if we consider the field φ, then the action of $T_{\mathrm{rot}}(g)$ is to shift the phase by $\frac{2\pi}{\omega}$:

$$T_{\mathrm{rot}}(g)\varphi T^{\dagger}_{\mathrm{rot}}(g) = e^{\frac{2\pi i}{\omega}}\varphi. \tag{3.27}$$

If we remember that $T^{\dagger}_{\mathrm{rot}}(g)T_{\mathrm{rot}}(g) = 1$ since the transformation is unitary, then by a simple symmetry argument it is straightforward to see that this leads to a vanishing expectation value for the φ with respect to the topological vacuum $|\Omega\rangle$:

$$\langle\Omega \mid \varphi \mid \Omega\rangle = \langle\Omega \mid T^{\dagger}_{\mathrm{rot}}(g)T_{\mathrm{rot}}(g)\ \varphi\ T^{\dagger}_{\mathrm{rot}}(g)T_{\mathrm{rot}}(g) \mid \Omega\rangle = e^{\frac{2\pi i}{\omega}}\langle\Omega \mid \varphi \mid \Omega\rangle = 0. \tag{3.28}$$

Similarly, for the reflection symmetry component of $\mathrm{Dih}(\omega)$ acts on the vacuum states like:

$$T_{\mathrm{ref}}(g)|N\rangle = T^{n\dagger}_{\mathrm{rot}}(g)PT^{n}_{\mathrm{rot}}(g)|N\rangle \implies T(g)|\Omega\rangle = A\sum_{N}T^{n\dagger}_{\mathrm{rot}}(g)PT^{n}_{\mathrm{rot}}(g)|N\rangle = |\Omega\rangle, \tag{3.29}$$

where P indicates the parity operator and T^{n}_{rot} means n sequential applications of the rotation generator. In the same way, for the scalar field φ:

$$T_{\mathrm{ref}}(g)\varphi T^{\dagger}_{\mathrm{ref}}(g) = T^{n\dagger}_{\mathrm{rot}}(g)PT^{n}_{\mathrm{rot}}(g)\varphi = e^{i\vartheta(\varphi)}\varphi, \tag{3.30}$$

for some angle $\vartheta(\varphi)$ that depends on the starting phase of the field φ. Therefore, we can use the same argument to show the vanishing expectation value:

$$\langle\Omega \mid \varphi \mid \Omega\rangle = \langle\Omega \mid T_{\mathrm{ref}}(g)T^{\dagger}_{\mathrm{ref}}(g)\ \varphi\ T_{\mathrm{ref}}(g)T^{\dagger}_{\mathrm{ref}}(g) \mid \Omega\rangle = e^{i\vartheta(\varphi)}\langle\Omega \mid \varphi \mid \Omega\rangle = 0. \tag{3.31}$$

These results confirm that at low energies, the expectation value $\langle\varphi\rangle$ vanishes due to symmetry restoration from instanton effects, resulting in $\langle\theta^{\mu\nu}\rangle = 0$. At high energies, where we are unable to construct a vacuum with such symmetry, $\langle\varphi^{a}\rangle$ becomes non-zero, leading to $\langle\theta^{\mu\nu}\rangle \neq 0$ and the emergence of Lorentz violation.

3.4 Linear Sigma Model

To further explore the modulation of the noncommutativity parameter $\theta^{\mu\nu}$, we propose an alternative model where the scalar fields φ^a are described collectively by a single Lagrangian—the Linear Sigma Model. This model possesses an $O(N)$ symmetry, where $N = \frac{d(d-1)}{2}$ and $d = \dim \Theta$, the dimension of the antisymmetric matrix space to which $\theta^{\mu\nu}$ belongs. For clarity and without loss of generality, we will consider $d = 3$, so $N = 3$, corresponding to an $O(3)$ symmetry. The conclusions drawn here can be generalized to any dimension d such as $d = 4$ which would be the correct dimensionality for MNCQFT. In this model, unlike the sine-Gordon model, we will not construct the explicit form of the topological vacuum $|\Omega\rangle$. Instead, we will show that the instantons in this model will modify the two-point correlation function of the theory. Then, the vanishing expectation value for the scalar fields φ^a will result from applying the cluster decomposition principle. To this aim, consider the Linear Sigma Model for $N = 3$ real scalar fields φ^i:

$$\begin{cases} \mathcal{L} = \dfrac{1}{2} \displaystyle\sum_{i=1}^{3} \left(\partial_\mu \varphi^i \partial^\mu \varphi^i \right) - V(\varphi^1, \varphi^2, \varphi^3), \\ V(\varphi^1, \varphi^2, \varphi^3) = -\dfrac{1}{2} \mu^2 \displaystyle\sum_{i=1}^{3} (\varphi^i)^2 + \dfrac{1}{4} \lambda \left(\displaystyle\sum_{i=1}^{3} (\varphi^i)^2 \right)^2 . \end{cases} \tag{3.32}$$

Here, μ and λ are constants that set the scale of spontaneous symmetry breaking and the strength of the self-interaction, respectively. In this case, the Linear Sigma Model requires that the scalar fields φ^i be real-valued functions of time:

$$\varphi^i(x) : \mathbb{R} \to \mathbb{R}. \tag{3.33}$$

Moreover, as is characteristic for the Linear Sigma model, the potential V is minimized when the fields satisfy:

$$\sum_{i=1}^{3} (\varphi^i)^2 = \frac{\mu^2}{\lambda}, \tag{3.34}$$

which defines a vacuum manifold with the topology of a sphere S^2, reflecting the $O(3)$ symmetry of the model.

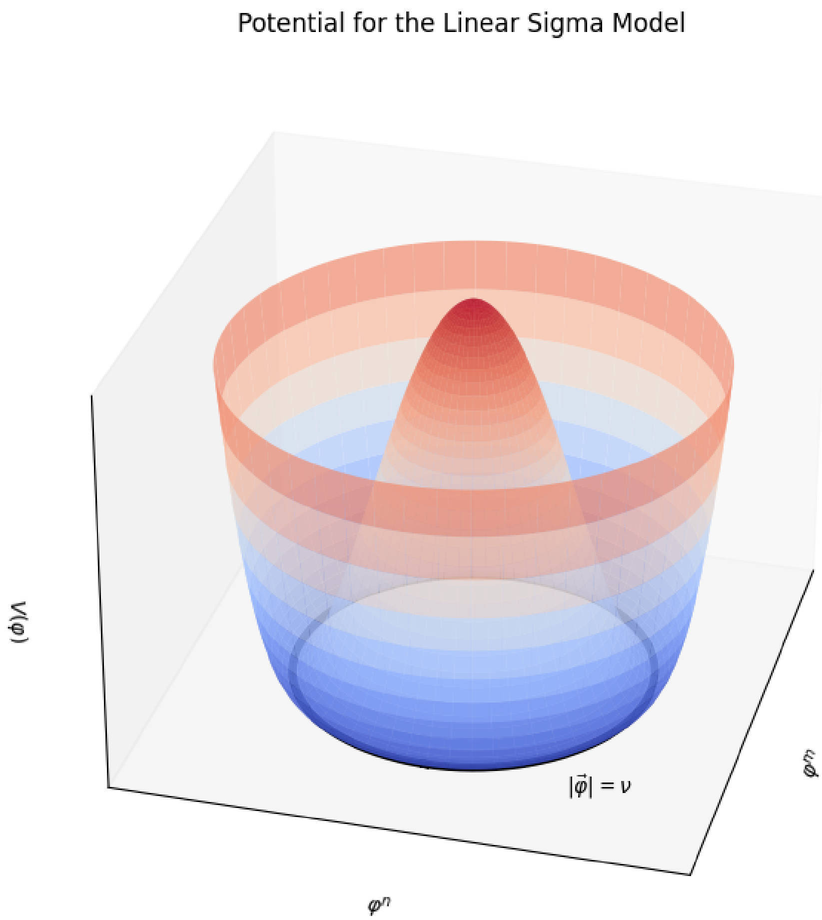

Figure 3.3: Potential for the Linear Sigma Model restricted to two fields.

Choosing a specific vacuum expectation value spontaneously breaks the $O(3)$ symmetry. Without loss of generality, we select the vacuum in which only the first component acquires a non-zero value:

$$\vec{\varphi} = (\nu, 0, 0), \quad \nu = \frac{\mu}{\sqrt{\lambda}}. \tag{3.35}$$

This choice simplifies the analysis by reducing the model to one involving only the field $\varphi \equiv \varphi^1$, while the other components correspond to Goldstone modes resulting from the symmetry breaking whose dynamics are uninteresting for the model at the moment. In this broken-symmetry state, when applying the ultralocal limit, its Euclidean Lagrangian is

$$\mathcal{L}_{\text{ul}} = \frac{1}{2}(\partial_\tau \varphi)^2 + \frac{\lambda}{4}\left(\varphi^2 - \nu^2\right)^2. \tag{3.36}$$

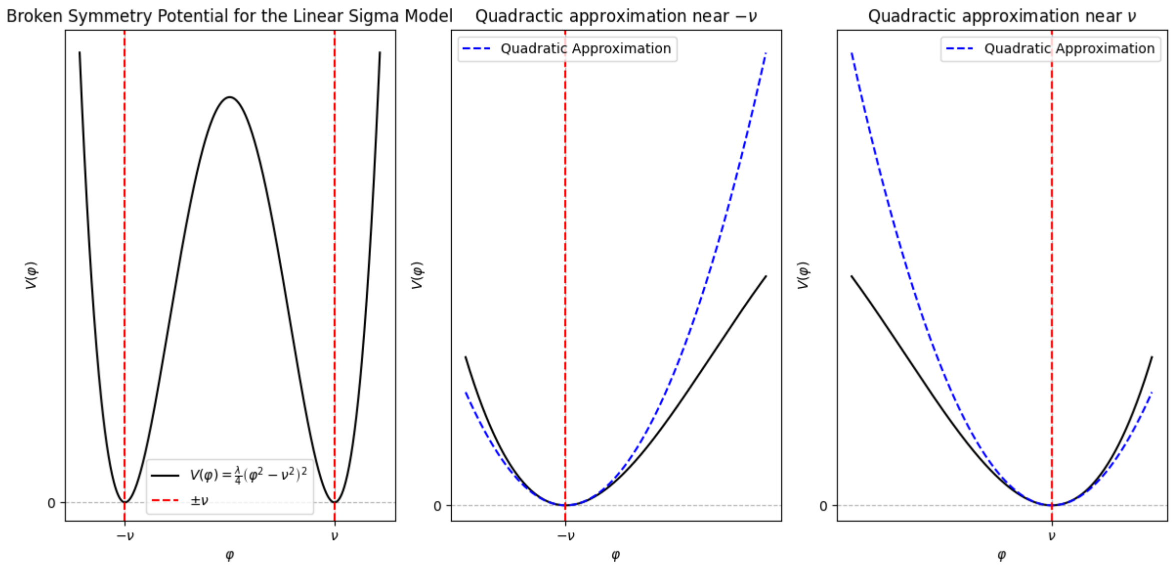

Figure 3.4: Potential for the Linear Sigma Model restricted to a single field after SSM with its quadratic approximation near the minima.

By imposing the boundary conditions corresponding to tunneling between the degenerate vacua:

$$\begin{cases} \lim_{\tau \to -\infty} \varphi(\tau) = -\nu, \\ \lim_{\tau \to +\infty} \varphi(\tau) = \nu. \end{cases} \tag{3.37}$$

Then, we can use the first Bogomolny equation to find the instanton solution:

$$\begin{aligned} \pm \int d\tau &= \sqrt{\frac{2}{\lambda}} \int \frac{d\varphi}{\varphi^2 - \nu^2} \\ \implies \pm\Delta\tau &= \sqrt{\frac{2}{\lambda}} \frac{1}{2\nu} \left| \ln \frac{\varphi - \nu}{\varphi + \nu} \right| \Bigg|_{\varphi} . \end{aligned} \tag{3.38}$$

Which implies that the instanton solution takes the form:

$$\frac{\varphi + \nu}{\varphi - \nu} = e^{\pm \mu \sqrt{2}(\tau - \tau')} \implies \overline{\varphi}(\tau) = \nu \tanh\left(\pm \frac{\mu(\tau - \tau')}{\sqrt{2}} \right), \tag{3.39}$$

where τ' is the time around which the instanton is centered, corresponding to the moduli space of instantons in this model. Moreover, in this case, the characteristic time of the instanton is $\tau_0 = \frac{\sqrt{2}}{\mu}$.

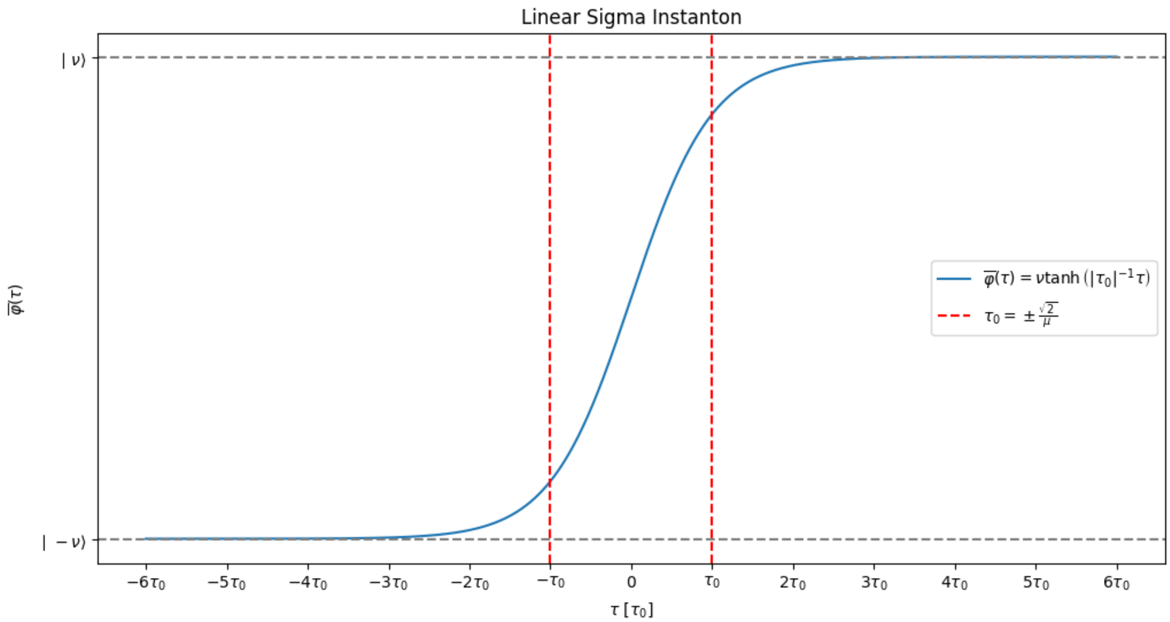

Figure 3.5: Single instanton for the Linear Sigma model.

The action for this instanton, like for the sine-Gordon model, is finite and corresponds directly to its energy. Using the action side of the second Bogomolny equation:

$$\begin{aligned} S[\overline{\varphi}] &= \int_{-\infty}^{\infty} d\tau \ \left[\frac{1}{2}(\partial_\tau\varphi)^2 + \frac{\lambda}{4}\left(\varphi^2-\nu^2\right)^2\right] \\ &= \frac{\lambda}{2}\int_{-\infty}^{\infty} d\tau(\varphi)\ (\varphi^2-\nu^2)^2 \\ &= \frac{2\lambda\nu}{\sqrt{2}\mu}\int_0^{\nu} d\varphi(\varphi^2-\nu) \\ &= \frac{2\sqrt{2}\mu^3}{3\lambda} = E[\overline{\varphi}]. \end{aligned} \tag{3.40}$$

Thus, the tunneling amplitude will be:

$$\langle\nu_+|\, e^{-\tau\mathcal{H}/\hbar}\,|\nu_-\rangle = K\Delta N e^{-\frac{2\sqrt{2}\mu^3}{3\lambda\hbar}} = \xi, \tag{3.41}$$

where, just like in the sine-Gordon model, K is the correction term resulting from approximating the vacuum states of the Linear Sigma model to the harmonic oscillator. Moreover, it is also possible to compute the amplitude for the transition $|\nu,0,0\rangle \rightarrow |0,\nu,0\rangle \rightarrow |0,0,\nu\rangle$, where a non-zero field transitions to a zero field and vice versa. This tunneling ensures that there is a full mixing of the vacuum states of the theory [13], this allows for the full restoration of the $O(3)$ symmetry of the theory.

Now, for this model it is worthwhile to compute the value of ΔN explicitly as we will require it to derive the two-point correlation function induced by instantons. Since we have chosen the model in such a way that we can approximate each of the vacuum states as a harmonic oscillator, we can write:

$$N\Delta = \left(\frac{V''(\pm\nu)}{\pi\hbar}\right)^{-\frac{1}{2}} e^{-V''(\pm\nu)/2}K. \tag{3.42}$$

Therefore, by identifying the components, we can retrieve the normalization constant as:

$$N = \left(\frac{V''(\pm\nu)}{\pi\hbar}\right)^{-\frac{1}{2}}. \tag{3.43}$$

Thus, the only value that we are missing is the correction factor K. If we recall, instantons possess a moduli space, a collective coordinate that characterizes instantons but whose value is unimportant for the theory since observables must be independent from it. For example, the action of the instanton is independent of this factor. The correction factor K must take into account integration over the moduli space of the instantons and remove the zero modes. Following a procedure analogous to the Faddeev-Popov gauge fixing procedure [11, 31], the correction factor is given by:

$$K = \left(\frac{S[\overline{\varphi}]}{2\pi\hbar}\right)^{\frac{1}{2}} \left|\frac{\det(-\partial_\tau^2 + V''(\pm\nu))}{\det(-\partial_\tau^2 + V''(\overline{\varphi}))}\right|\Bigg|_{\lambda_{\overline{\varphi}}\neq 0} = \left(\frac{S[\overline{\varphi}]}{2\pi\hbar}\right)^{\frac{1}{2}} \zeta(\pm\nu, \overline{\varphi}), \tag{3.44}$$

which is computed by excluding all zero-modes $\lambda_{\overline{\varphi}} = 0$ from the determinant.

3.4.1 Multi-instantons and Dilute Gas Approximation

By analyzing the moduli space of the instantons, we find that, in reality, we should consider the effects of instantons characterized by a wide set of times τ_n, the collective coordinate. Thus, we postulate that for a multi-instanton of length n, the correction factor $K \to K^n$, such that:

$$N\Delta = \left(\frac{V''(\pm\nu)}{\pi\hbar}\right)^{-\frac{1}{2}} e^{-V''(\pm\nu)/2}K^n. \tag{3.45}$$

Considering the integration over the moduli space, we compute:

$$\int_{\tau_1 < \ldots < \tau_n} d\tau_1 \ldots d\tau_n = \frac{T^n}{n!}. \tag{3.46}$$

Thus, the tunneling amplitude is modified to account for infinite length multi-instanton chains [31]:

$$\begin{aligned} \langle \nu_\pm | \, e^{-\tau \mathcal{H}/\hbar} \, | \nu_- \rangle &= \Delta N \sum_{n \text{ odd/even}} \frac{\left(K e^{-S[\overline{\varphi}]]/\hbar} T \right)^n}{n!} \\ &= \Delta N \frac{1}{2} \left(e^{K \exp(-S[\overline{\varphi}] T/\hbar)} \mp e^{-K \exp(-S[\overline{\varphi}] T/\hbar)} \right). \end{aligned} \tag{3.47}$$

So that for the dilute gas approximation:

$$\begin{cases} \langle \nu_- | \, e^{-\tau \mathcal{H}/\hbar} \, | \nu_- \rangle = \Delta N \cosh \left(e^{K \exp(-S[\overline{\varphi}] T/\hbar)} \right), \\ \langle \nu_+ | \, e^{-\tau \mathcal{H}/\hbar} \, | \nu_- \rangle = \Delta N \sinh \left(e^{K \exp(-S[\overline{\varphi}] T/\hbar)} \right). \end{cases} \tag{3.48}$$

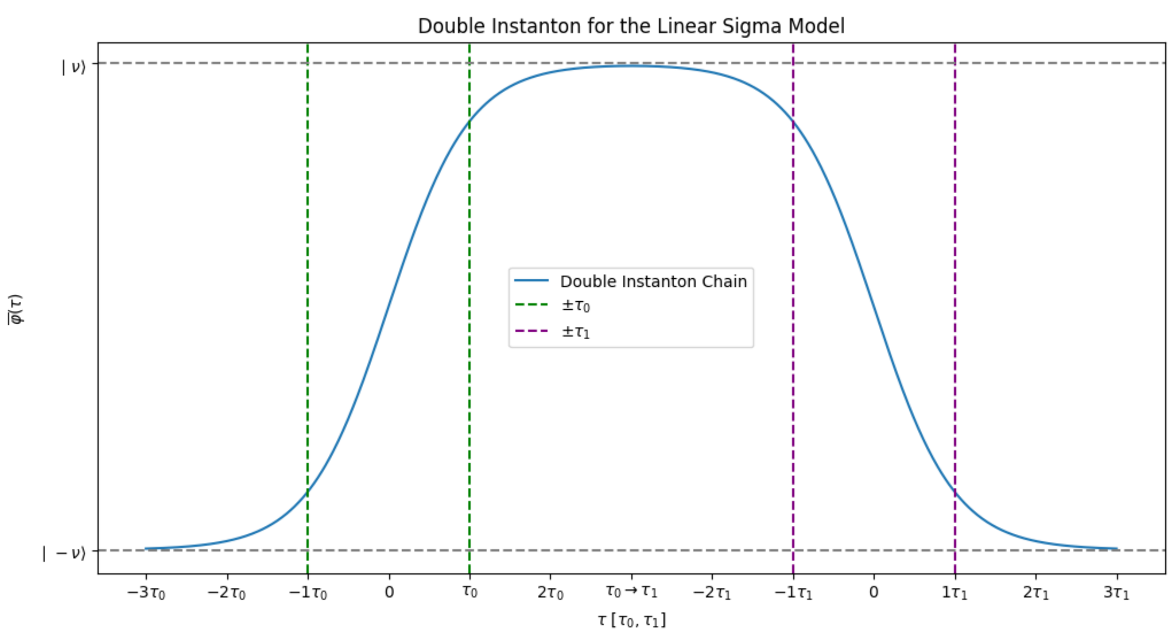

Figure 3.6: Double instanton for the Linear Sigma model.

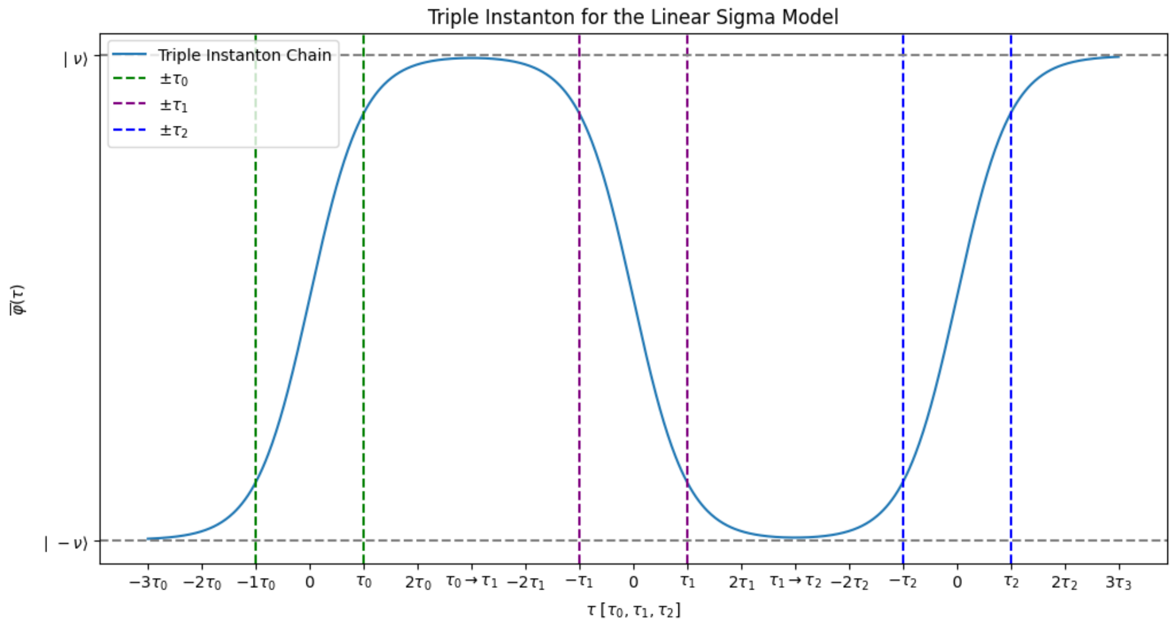

Figure 3.7: Triple instanton for the Linear Sigma model.

3.4.2 Two-Point Correlation Function, Cluster Decomposition and Vanishing Expectation Value

Unlike the sine-Gordon model, we will not construct the explicit form of the topological vacuum $|\Omega\rangle$, as the following analysis is independent of this form. The two-point correlation function in the dilute gas approximation is given by [11]:

$$\langle \varphi(0)\varphi(\tau)\rangle = \frac{\mu^2}{\lambda} e^{-2KS[\overline{\varphi}]\exp(-S[\overline{\varphi}])\tau} = \frac{\mu^2}{\lambda} e^{-\Delta E \tau}, \tag{3.49}$$

where $\Delta E = 2KS[\overline{\varphi}]\exp(-S[\overline{\varphi}])$ is the energy difference between the ground state and the first excited state. As $\tau \to \infty$, the correlation function vanishes:

$$\lim_{\tau\to\infty} \langle \varphi(0)\varphi(\tau)\rangle = 0. \tag{3.50}$$

Applying the Cluster Decomposition Principle [17], this result implies:

$$\langle \varphi \rangle = 0. \tag{3.51}$$

This vanishing expectation value arises due to the restoration of symmetry by instanton effects, despite spontaneous symmetry breaking at the classical level. At high energies, the ultralocal limit no longer applies because $E^2 \ll \mu^2$ is violated, so instanton effects vanish by the Derrick-Hobart theorem, recovering the desired behavior. For this model, the explicit energy cutoff is $\Lambda \approx \mu$.

The validity of the cluster decomposition principle under ultralocal dimensionality restrictions will be discussed further in the next section. By carefully choosing the coupling constants μ and λ, compatibility between these results and the ultralocal framework is achieved.

3.5 Implications for the Modulation of $\theta^{\mu\nu}$

As we have seen, with both models—the sine-Gordon model and the Linear Sigma Model—the vanishing expectation value of φ at low energies leads to a vanishing expectation value of the noncommutativity parameter $\theta^{\mu\nu}$:

$$\langle\theta^{\mu\nu}\rangle = \sum_a \langle\varphi^a\rangle\theta^a = 0. \tag{3.52}$$

At low energies ($E < \Lambda$), the effects of instantons allow for the Lorentz symmetry to remain unbroken, hence the theory effectively reduces to the Standard Model. However, at energies above a certain cutoff Λ, the expectation value $\langle\varphi^a\rangle$ can become non-zero due to the suppression of instanton effects, leading to:

$$\langle\theta^{\mu\nu}\rangle \neq 0. \tag{3.53}$$

Explicitly, this behavior at high energies arises from our inability to construct the topological vacuum in the sine-Gordon model and the inapplicability of the ultralocal limit in the Linear Sigma Model. This non-zero expectation value introduces Lorentz-violating terms in the theory, consistent with the behavior expected in a NCQFT with SLIV. Thus, we recover the desired behavior:

$$\langle\varphi^a\rangle \begin{cases} \langle\varphi^a\rangle_{E<\Lambda} = 0 \implies \langle\theta^{\mu\nu}\rangle = 0, \\ \langle\varphi^a\rangle_{E>\Lambda} \neq 0 \implies \langle\theta^{\mu\nu}\rangle \neq 0. \end{cases} \tag{3.54}$$

This energy-dependent behavior of $\theta^{\mu\nu}$ ensures that Lorentz violation is absent at low energies, in agreement with experimental observations, while allowing for Lorentz-violating effects to emerge at high energies. This can be seen if we recall the Seiberg-Witten Lagrangian for NCQED:

$$\begin{aligned}\mathcal{L}_{\text{QED, SW}} =& \frac{1}{2} i\bar{\psi}\gamma^{\mu} D_{\mu}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &- \frac{1}{8} iq\theta^{\alpha\beta}F_{\alpha\beta}\bar{\psi}\gamma^{\mu}D_{\mu}\psi + \frac{1}{4}iq\theta^{\alpha\beta}F_{\alpha\mu}\bar{\psi}\gamma^{\mu}D_{\beta}\psi \\ &+ \frac{1}{4}mq\theta^{\alpha\beta}F_{\alpha\beta}\bar{\psi}\psi \\ &- \frac{1}{2}q\theta^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu} + \frac{1}{8}q\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\theta^2) \\ &= \mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{LIV}}(\theta).\end{aligned} \tag{3.55}$$

Then, below the energy cutoff Λ, we have $\langle\theta^{\mu\nu}\rangle = 0$, so that our Lagrangian reduces to:

$$\mathcal{L}_{\text{QED, SW, } E<\Lambda} = \mathcal{L}_{\text{QED}} \ , \tag{3.56}$$

which is manifestly Lorentz invariant. On the other hand, above the energy cutoff Λ, we have $\langle\theta^{\mu\nu}\rangle \neq 0$, so that:

$$\mathcal{L}_{\text{QED, SW, } E>\Lambda} = \mathcal{L}_{\text{QED, SME}} \ , \tag{3.57}$$

which is not Lorentz invariant. This behavior is the characteristic signature of a SLIV theory. Thus, the required behavior to implement the model within Kostelecky's framework is achieved [4, 6, 23].

Chapter 4

Consistency, Implications and Limitations

4.1 Compatibility with Ultralocal Limit

Ensuring compatibility with the ultralocal limit is essential for maintaining the stability and consistency of solutions in field theories like the Linear Sigma Model, particularly when attempting to bypass the Derrick-Hobart theorem and addressing the issues related to tunneling in infinite space. While the sine-Gordon model naturally satisfies these requirements due to the compactification of space into S^1, the Linear Sigma Model demands a more thorough analysis. This section outlines the necessary spatial and temporal constraints for achieving the ultralocal limit.

To apply the ultralocal limit effectively, we must impose constraints on the spatial and temporal variations of the field. Specifically, the characteristic sizes of these variations are represented by:

$$\frac{1}{T} \sim \left|\frac{\partial_t \varphi}{\varphi}\right|, \quad \frac{1}{L} \sim \left|\frac{\nabla \varphi}{\varphi}\right|. \tag{4.1}$$

These expressions quantify the inverse time scale T and inverse length scale L over which significant changes in the field φ occur. For the ultralocal limit to be valid, we require that temporal variations dominate over spatial variations, ensuring that:

$$\frac{1}{T} \gg \frac{1}{L} \quad \text{or equivalently} \quad T \ll L. \tag{4.2}$$

This condition implies that the characteristic time scale of changes in the field is much shorter than the characteristic spatial scale. Physically, this ensures that the field configuration varies more rapidly in time than in space, effectively suppressing spatial gradients and emphasizing the temporal dynamics. This dominance of temporal changes is critical for avoiding the Derrick-Hobart theorem. Thus, by neglecting spatial variation, we effectively reduce the dimensionality of the theory, bypassing the Derrick-Hobart Theorem. The mechanism for achieving the ultralocal limit is the insertion of a infrared cutoff to spatial variation such that when:

$$\ell \ll \tau_0^{-1} < T \ll L < \infty \implies \mathcal{L}_{\text{eff, kin}} \to \mathcal{L}_{\text{ul, kin}}, \tag{4.3}$$

and the theory admits instatons with characteristic time τ_0^{-1}. As we can see, this inequality can always be satisfied since ℓ is a free parameter. Thus, we are also justified in applying the ultralocal limit to the Linear Sigma Model. The problem arises when we consider that for the vanishing of the expectation value of the scalar field φ^a we require to further satisfy the Dilute Gas Appoximation and the Cluset Decomposition Principle requirements.

4.1.1 Compatibility with Dilute Gas Approximation and Cluster Decomposition Principle in the Ultralocal Limit

Beyond the ultralocal limit conditions, the model must also satisfy the requirements for the Dilute Gas Approximation and the Cluster Decomposition Principle. For the Dilute Gas Approximation to be valid, the time separation between instantons, τ, must be significantly larger than the characteristic time of an individual instanton, τ_0. This condition is expressed as:

$$\tau_0 \sim \frac{\sqrt{2}}{\mu} < \tau. \tag{4.4}$$

This requirement ensures that individual instantons are well-separated in time, preventing overlap and allowing the approximation of a dilute gas of instantons. The corresponding time scale condition becomes:

$$\frac{\sqrt{2}}{\mu} \ll T \ll L. \tag{4.5}$$

Here, T must be long enough to accommodate multiple well-separated instanton events but still bounded by L to prevent spatial fluctuations from dominating.

The Cluster Decomposition Principle necessitates that the time separation τ between field configurations be large enough so that the two-point correlation function in the time domain vanishes at large separations. This ensures statistical independence of distant events, which is critical for the physical interpretation of correlation functions. The condition for this principle is:

$$\frac{\sqrt{2}}{\mu} \ll \Delta E^{-1} \ll \tau, \tag{4.6}$$

where ΔE was the energy difference between the ground state and the first excited state for the Linear Sigma Model used to compute the time-only two-point correlation function. It is noteworthy that ΔE depends on the two arbitrary coupling constants of the model, μ and λ. For the Linear Sigma Model, the final conditions take the form:

$$\frac{\sqrt{2}}{\mu} \ll \Delta E^{-1} \ll T \ll L < \infty. \tag{4.7}$$

Due to the dependence of ΔE on the coupling constants, we have enough degrees of freedom to ensure that this inequality is satisfied by appropriately tuning the coupling constants. Specifically, the relationship between μ and λ can be expressed as:

$$\frac{\sqrt{2}}{\mu} \ll (2KS[\overline{\varphi}]\exp(-S[\overline{\varphi}]))^{-1} \implies \zeta\sqrt{2}\left(\frac{8\sqrt{2}\mu^3}{3\pi\hbar\lambda}\right)^{\frac{1}{2}} \exp\left(\frac{-2\sqrt{2}\mu^3}{3\lambda}\right) \ll \mu, \tag{4.8}$$

where ζ is the factor obtained when computing the correction factor K for the harmonic oscillator approximation. Moreover, the flexibility in adjusting these constants ensures that the model can simultaneously meet the requirements of the ultralocal limit, the Dilute Gas Approximation, and the Cluster Decomposition Principle.

Figure 4.1: Inequality satisfaction in Log domain for the consistency requirement in terms of the coupling constant.

4.2 Consistency of the Model and Its Limitations

4.2.1 Consistency with the Seiberg-Witten Map and Kostelecký's Framework

By assuming that $\varphi^a(t)$ depends solely on time and neglecting spatial variations, we ensure that $\theta^{\mu\nu}$ remains effectively constant within local spatial neighborhoods at any fixed time, by considering the smearing of observables [19, 20] to appear constant within the causal neighborhood. This allows us to apply the Seiberg-Witten map at a fixed energy scale, with $\theta^{\mu\nu}$ treated as a constant parameter during the mapping process.

The absence of interactions between the modulating scalar fields and SM fields reinforces the validity of the Seiberg-Witten map in this context. Since the scalar fields influence only $\theta^{\mu\nu}$ and do not introduce additional gauge or matter couplings, the fundamental assumptions underlying the map remain intact. This separation of the moduli space from Minkowski spacetime preserves the gauge invariance and locality essential for the Seiberg-Witten map's applicability.

Moreover, another aspect that supports the compatibility of the model with the Seiberg-Witten map relies on the fact that, even if some anomalous behaviors were to be introduced by the model as the noncommutative parameter depends on spacetime that we have not taken into account in this analysis. We can still apply Kontsevich's quantization procedure. This quantization procedure is a generalization of the Weyl-Wigner quantization used for constructing NCQFT. This quantization procedure can be done for any arbitrary Poisson structures [1, 3, 22]. Hence, this model would still allow us to construct a quantum field theory. Moreover, in a work by Aschieri it has also been shown that we can construct the Seiberg-Witten map for any quantum theory constructed by following Kontsevich's procedure [1]. Thereafter, since the Seiberg-Witten map depends only on the gauge orbits of the theory and the target theory, and this model has not introduced any new gauge symmetry, the result from applying must be the same SME described by Caroll [1, 9]. This SME being compatible with Kostelecký's framework.

Moreover, within Kostelecký's framework, to show the conservation of the stress-energy tensor in Riemann-Cartan spacetime in theories with SLIV, it is paramount that $\theta^{\mu\nu}$ remain constant within a causal neighborhood. We postulate that our ability to reconstruct the spacetime field $\varphi(\chi, t)$ is paramount for the consistency of the model proposed. For example, by considering a constant smearing function f:

$$f(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases} \tag{4.9}$$

This would be inherited to the smeared field $\varphi^a(\chi, t)$ and in turn to $\theta^{\mu\nu}(\chi, t)$ which would behave as a constant within Ω, a causal neighborhood. Thus allowing for consistency with Kostelecký's results. Moreover, even if our postulate produced undesired effects. It is noteworthy to state that Kostelecký postulates that the requirement for the coefficients for Lorentz violation to be constant in a neighborhood could be dropped without affecting the conservation laws. As long as the dynamical coefficients have integral curves or sufficiently slow spacetime variation, where constancy can be assumed as the leading approximation [23]. Thus, in both cases our model should remain consistent with Kostelecký's framework.

4.2.2 Addressing Potential Limitations

While the model presents a coherent framework, certain limitations require attention. The assumption that the scalar fields do not interact with SM fields simplifies the model but may need to be examined for potential indirect effects. For instance, fluctuations in $\theta^{\mu\nu}$ could, in principle, influence the renormalization of SM fields or lead to observable consequences that need to be accounted for. Most notably, by the effect of a SLIV theory, we introduce an effective cutoff, ultraviolet for the SM sector of the SME and infrared for the LIV sector. This effects need to be explored further as to avoid possible incompatibilities with current scientific concensus regarding renormalization.

Furthermore, the assumption that the scalar fields' dynamics are confined to the moduli space implies that their equations of motion do not influence Minkowski spacetime. Investigating whether this separation can be consistently maintained while still allowing for the modulation of spacetime structure is paramount for the consistency of this model and should be explored further. Most notably, the ultralocal requirement necessary for the existence of instantons in the proposed models implies a strict requirement between time T and space L fluctuations in the moduli space. Meanwhile, T and L behave normally on Minkowski space. It is important to explore the implications of these restricted scales on Minkowski spacetime, specially, their effects on the phenomenological signatures of the SM.

Another important aspect that should be explored further lies within higher-order effects. Most notably, although this model has a vanishing expectation value of the noncommutativity parameter, it is only shown at first order in θ. Further investigation should be done to explore the effects of higher-order terms that may not have a vanishing expectation value.

Potential phenomenological implications, such as deviations from Lorentz invariance at accessible energy scales, must be carefully analyzed. Even if the effects are suppressed at low energies, precision experiments that have been done place stringent constraints on the model [4]. Assessing these implications will help determine the model's viability and guide any necessary refinements. Carroll proposes several experiments that could be done and mentions some of these constrains [4].

Chapter 5

Conclusion

This work has introduced a novel framework for dynamically modulating the noncommutativity parameter $\theta^{\mu\nu}$ in NCQFT, enabling the incorporation of SLIV. The key idea was to tie $\theta^{\mu\nu}$ to the dynamics of scalar fields φ^a residing in a moduli space external to Minkowski spacetime. By allowing $\theta^{\mu\nu}$ to evolve with energy, the model achieves a seamless transition between Lorentz-invariant behavior consistent with the Standard Model at low energies and Lorentz-violating effects at high energies described by a Standard Model Extension (SME).

The modulation of $\theta^{\mu\nu}$ was achieved by expressing it as a linear combination of antisymmetric matrices θ^a, weighted by the scalar fields φ^a. This approach ensures that the behavior of $\theta^{\mu\nu}$ is governed entirely by the dynamics of the scalar fields, which vary as a function of energy. At low energies, the vacuum expectation value of the scalar fields vanishes due to the effects of instantons, leading to $\langle\theta^{\mu\nu}\rangle = 0$ and the recovery of Lorentz invariance. At high energies, where instanton effects are suppressed, the scalar fields acquire non-zero expectation values, yielding $\langle\theta^{\mu\nu}\rangle \neq 0$ and introducing Lorentz-violating terms into the theory. This modulation is consistent with the structure of SMEs providing a natural way to incorporate SLIV in NCQFT.

Two scalar field models were proposed to implement this mechanism: the sine-Gordon model and the Linear Sigma model. These models were chosen for their well-understood instanton dynamics and their ability to produce the desired energy-dependent behavior of φ^a. The sine-Gordon model employs a periodic potential that admits nontrivial topological vacua, allowing for instanton-mediated transitions that restore symmetry at low energies. The Linear Sigma Model, on the other hand, relies on spontaneous symmetry breaking to produce degenerate vacua, with instanton solutions ensuring a vanishing expectation value for the scalar fields at low energies by the Cluster Decomposition Principle. In both cases, the models satisfy the conditions required for the ultralocal limit, which suppresses spatial fluctuations and emphasizes temporal dynamics.

The ultralocal limit played a central role in the theoretical construction of the model, providing a way to circumvent the Derrick-Hobart theorem, which prohibits stable, finite-energy solutions in scalar field theories in dimensions $n \geq 2$. By suppressing spatial variations and focusing on temporal fluctuations, the ultralocal limit effectively reduces the dimensionality of the problem, allowing for the existence of instantons. The constraints on spatial and temporal scales required for the ultralocal limit were derived explicitly, ensuring compatibility with the dilute gas approximation and the cluster decomposition principle. These conditions ensured that instanton dynamics remained consistent with the assumptions of the model, even in the presence of multiple instantons or tunneling events.

The compatibility of the model with the Seiberg-Witten map and Kostelecký's framework was also established. In this model, the ultralocal limit ensures that $\theta^{\mu\nu}$ is effectively constant within causal neighborhoods, allowing the Seiberg-Witten map to be applied without modification. Moreover, by situating the scalar fields in a moduli space external to Minkowski spacetime, their dynamics do not interfere with the interactions of Standard Model fields nor they introduce new gauge symmetries, thus ensuring that the NCQFT is mapped to the correct SME. That is, the proposed models remain consistent with Kostelecký's results on the compatibility of SLIV theories with Einstein-Cartan spacetimes.

While the model provides a coherent framework for incorporating SLIV in NCQFT, certain limitations and challenges require further investigation. Higher-order corrections to $\theta^{\mu\nu}$ have not been explored in this work. While the expectation value of $\theta^{\mu\nu}$ vanishes at low energies to first order, it is possible that higher-order terms could introduce nontrivial effects that need to be analyzed. The renormalization of the theory, particularly the interplay between the ultraviolet cutoff for Standard Model fields and the infrared cutoff for Lorentz-violating terms, also presents a challenge. Ensuring compatibility with the well-established renormalization framework in quantum field theory will be crucial for the model's robustness.

Experimental constraints on Lorentz invariance violation also impose significant restrictions on the model. While the effects of $\theta^{\mu\nu}$ are suppressed at low energies, precision experiments in particle physics, astrophysics, and cosmology provide stringent bounds on any deviations from Lorentz invariance. Carroll and Kostelecký have proposed several experimental tests and outlined the constraints on SME parameters, which must be satisfied by this model. Future work should explore the phenomenological implications of the model, identifying specific experimental signatures that could confirm or falsify its predictions.

The insights gained from this work suggest several promising directions for future research. Extending the analysis to higher-dimensional scalar field models or exploring alternative mechanisms for symmetry breaking could provide additional insights into the behavior of $\theta^{\mu\nu}$ at different energy scales. Investigating the effects of higher-order corrections and quantum fluctuations on the scalar fields and their coupling to $\theta^{\mu\nu}$ could refine the model further. Exploring potential experimental signatures of SLIV, such as deviations in particle dispersion relations or anisotropies in high-energy processes, would provide a concrete way to test the model's predictions as well as, set definitive bounds of the effective energy cutoff Λ. Finally, generalizing the framework to include modulation of other dynamical parameters, such as gauge fields or coupling constants, could expand its applicability to a broader range of physical phenomena.

Bibliography

[1] P. Aschieri and L. Castellani. “Noncommutative gauge and gravity theories and geometric Seiberg–Witten map”. In: *The European Physical Journal Special Topics* 232.23–24 (2023), pp. 3733–3746.

[2] J. Barbón. *Introduction to noncommutative field theory*. Tech. rep. Procedings of the Summer School on Particle Physics, Trieste (2001), 2002.

[3] F. Bayen and M. Flato. “Deformation theory and quantization. I. Deformations of symplectic structures”. In: *Annals of Physics* 111.1 (1978), pp. 61–110.

[4] S. Carroll and J. Harvey. “Noncommutative Field Theory and Lorentz Violation”. In: *Physical Review Letters* 87.14 (2001).

[5] S. Coleman. “The Uses of Instantons”. In: *The Whys of Subnuclear Physics*. Ed. by A. Zichichi. Springer US, 1979, pp. 805–941.

[6] D. Colladay and V. Kostelecký. “Lorentz-violating extension of the standard model”. In: *Physical Review D* 58.11 (1998).

[7] A. Connes. *Noncommutative geometry*. Springer, 1994.

[8] G. van Dijk et al. *On the characterization of trace class representations and Schwartz operators*. 2015. arXiv: 1512.02451 [math.RT]. URL: https://arxiv.org/abs/1512.02451.

[9] M. Douglas and N. Nekrasov. “Noncommutative field theory”. In: *Reviews of Modern Physics* 73.4 (Nov. 2001), pp. 977–1029. ISSN: 1539-0756. DOI: 10.1103/revmodphys.73.977. URL: http://dx.doi.org/10.1103/RevModPhys.73.977.

[10] A. Dueñas-Vidal. “Teorías de Yang-Mills en espacios no conmutativos”. Unpublished manuscript. 2010.

[11] E. Fradkin. *Quantum Field Theory*. Princeton University Press, 2021.

[12] G. Francisco. "Quasiclassical approximation for ultralocal scalar fields". In: *Journal of Physics A: Mathematical and General* 19.4 (Mar. 1986), p. 503. DOI: 10.1088/0305-4470/19/4/012. URL: https://dx.doi.org/10.1088/0305-4470/19/4/012.

[13] J. Gamboa and J. López-Sarrión. "Nonperturbative aspects of spontaneous symmetry breaking". In: *International Journal of Modern Physics A* 36.13 (May 2021), p. 2150074. ISSN: 1793-656X. DOI: 10.1142/s0217751x21500743. URL: http://dx.doi.org/10.1142/S0217751X21500743.

[14] J. Gleason. "The C*-Algebraic Formalism of Quantum Mechanics". Unpublished manuscript. Aug. 2009. URL: https://math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Gleason.pdf.

[15] M. Hayakawa. "Perturbative analysis on infrared aspects of noncommutative QED on $\mathbb{R}^4$". In: *Physics Letters B* 478.1–3 (2000), pp. 394–400.

[16] Z. Huang, X. Wang, and C. Wang. "Generalized operators and operator-valued distributions in quantum field theory". In: *Acta Mathematica Scientia. Series B. (English Edition)* 23 (Apr. 2003). DOI: 10.1016/S0252-9602(17)30218-7.

[17] T. Inoue. "Instantons and Restoration of Symmetries". In: *Prog. Theor. Phys.* 59 (1978), p. 986. DOI: 10.1143/PTP.59.986.

[18] B. Jurčo, P. Schupp, and J. Wess. "Nonabelian noncommutative gauge theory via noncommutative extra dimensions". In: *Nuclear Physics B* 604.1–2 (June 2001), pp. 148–180. ISSN: 0550-3213. DOI: 10.1016/s0550-3213(01)00191-2. URL: http://dx.doi.org/10.1016/S0550-3213(01)00191-2.

[19] J. Klauder. "Ultralocal Quantum Field Theory". In: *Concepts in Hadron Physics*. Ed. by Paul Urban. Springer Vienna, 1971, pp. 227–276. ISBN: 978-3-7091-8284-0.

[20] J. Klauder. "Ultralocal Scalar Field Models". In: *Commun. Math. Phys.* 18 (1970), pp. 307–318. DOI: 10.1007/BF01649449.

[21] J. Klaus. *Topology*. Springer Internation Publishing, 1984.

[22] M. Kontsevich. "Deformation Quantization of Poisson Manifolds". In: *Letters in Mathematical Physics* 66.3 (Dec. 2003), pp. 157–216. ISSN: 0377-9017. DOI: 10.1023/b:math.0000027508.00421.bf. URL: http://dx.doi.org/10.1023/B:MATH.0000027508.00421.bf.

[23] A. Kostelecký. "Gravity, Lorentz Violation, and the Standard Model". In: *Physics Review* 378.4 (2003), pp. 207–299.

[24] G. Landi. *An introduction to noncommutative spaces and their geometries*. Vol. 51. Springer Science & Business Media, 2003.

[25] N. Manton. *Topological Solitons*. Cambridge University Press, 2004.

[26] M. Paranjape. *The Theory and Applications of Instanton Calculations*. Cambridge University Press, 2022.

[27] R. Rajaraman. *An Introduction to Solitons and Instantons in Quantum Field Theory*. Elsevier Science Publishers, 1989.

[28] M. Rieffel. *Deformation Quantization for Actions of* $\mathbb{R}^d$. American Mathematical Soc., 1993.

[29] N. Seiberg and E. Witten. "Non-commutative Geometry and String theory". In: *JHEP* 9909 (1999), p. 032.

[30] R. Szabo. "Quantum field theory on noncommutative spaces". In: *Physics Reports* 378.4 (2003), pp. 207–299.

[31] F. Tanedo. "Instantons and their applications". Candidacy Work. Cornell University, May 2010.

[32] A. Teta. *A Mathematical Primer on Quantum Mechanics*. Springer International Publishing, 2018.