UNIVERSIDAD SAN FRANCISCO DE QUITO USFQ

Colegio de Posgrados

The Fractional Einstein Equations

Proyecto de investigación

Miguel Francisco Mena Larrea

Ernesto Contreras, Ph.D.
Director de Trabajo de Titulación

Nicola di Teodoro, Ph.D.
Codirector de Trabajo de Titulación

Trabajo de titulación presentado como requisito
para la obtención del titulo de Magíster en Física

Quito, 22 de diciembre de 2024

UNIVERSIDAD SAN FRANCISCO DE QUITO USFQ

Colegio de Posgrados

HOJA DE APROBACIÓN DE TRABAJO DE TITULACIÓN

The Fractional Einstein Equations

Miguel Francisco Mena Larrea

Nombre del Director de Programa:	Darío Niebieskikwiat
Título académico:	Doctor en Física
Director del programa de:	Maestría en Física

Nombre del Decano del Colegio Académico:	Eduardo Alba
Título académico:	Doctor en Ciencias Matemáticas
Decano del Colegio:	Colegio de Ciencias e Ingeniería

Nombre del Decano del Colegio de Posgrados:	Darío Niebieskikwiat
Título académico:	Doctor en Física

Quito, 22 de diciembre de 2024

© Derechos de Autor

Nombres y apellidos:	Miguel Francisco Mena Larrea
Código:	00335266
Cédula de Identidad:	1105233777
Lugar y fecha:	Quito, 22 de diciembre de 2024

ACLARACIÓN PARA LA PUBLICACIÓN

Nota: El presente trabajo, en su totalidad o cualquiera de sus partes, no debe ser considerado como una publicación, incluso a pesar de estar disponible sin restricciones a través de un repositorio institucional. Esta declaración se alinea con las prácticas y recomendaciones presentadas por el Committee on Publication Ethics COPE descritas por Barbour et al. (2017) Discussion document on best practice for issues around theses publishing, disponible en `http://bit.ly/COPETheses`

UNPUBLISHED DOCUMENT

Note: The following capstone project is available through Universidad San Francisco de Quito USFQ institutional repository. Nonetheless, this project – in whole or in part – should not be considered a publication. This statement follows the recommendations presented by the Committee on Publication Ethics COPE described by Barbour et al. (2017) Discussion document on best practice for issues around theses publishing available on `http://bit.ly/COPETheses`

Acknowledgments

I want to express my heartfelt gratitude to my parents for their unwavering support; a special thanks to Belén Loaiza, for her love and encouragement throughout this process.

I am also profoundly grateful to my thesis advisors, Ernesto Contreras and Nicola di Teodoro, whose guidance and insights were instrumental in bringing this project to fruition.

Lastly, I extend my appreciation to everyone in the Master in Physics program at USFQ. The past two years have been an incredible experience, both academically and personally. I will always cherish the memories and lessons learned during my time at this amazing institution.

Resumen

En este trabajo se investiga las ecuaciones de campo de Einstein (EFE) utilizando una nueva derivada fraccionaria que modifica el operador de Riemann-Liouville, estableciendo un marco consistente para el cálculo fraccionario en relatividad general. La derivada propuesta resuelve problemas asociados con el operador de Caputo y Riemann-Liouville estándar. Derivamos las EFE fraccionarias en espacio-tiempos de $2+1$ y $3+1$ dimensiones, asumiendo métricas estáticas y con simetrías circular y esférica. Como caso de estudio, evaluamos la solución del agujero negro de Bañados-Teitelboim-Zanelli (BTZ), mostrando que para parámetros fraccionarios cercanos a uno, corresponde a un agujero negro BTZ cargado con una constante cosmológica anisotrópica. Además, en cosmología examinamos soluciones con factores de escala polinomiales, relevantes para épocas dominadas por materia y radiación, discutiendo las implicaciones para la densidad de energía, la presión y la ecuación de estado en el contexto de modificaciones fraccionarias.

Palabras clave: *Derivada fraccionaria, Relatividad General, Agujero negro BTZ, Cosmología fraccionaria.*

Abstract

In this study, we investigate Einstein's field equations (EFE) using a novel fractional derivative that modifies the Riemann-Liouville operator to ensure a consistent framework for fractional calculus in General Relativity. The proposed derivative resolves issues with the Caputo and the standard Riemann-Liouville derivative. We derive fractional EFE in both $2+1$ and $3+1$ dimensional spacetimes, assuming static and circularly symmetric metrics. As a case study, we evaluate the Bañados-Teitelboim-Zanelli (BTZ) black hole solution, showing that for fractional parameters close to one, it corresponds to a charged BTZ black hole with an anisotropic cosmological constant. Additionally, in Cosmology, we examine solutions with polynomial scale factors relevant to matter- and radiation-dominated epochs, discussing the implications for energy density, pressure, and the equation of state in the context of fractional modifications.

Keywords: *Fractional derivative, General Relativity, BTZ black hole, Fractional Cosmology.*

Índice general

Índice de cuadros

Índice de figuras

Capítulo 1

Introduction

In recent years, fractional calculus has gained significant traction across various fields due to the added flexibility it provides in modeling the dynamics of complex physical phenomena as it introduces memory effect, which contrasts with the local operators used in classic theories. This non-locality has made fractional calculus particularly attractive for applications in a wide range of areas, including gravitational physics, where it offers the potential for a renormalizable model of quantum gravity [1–4]. A natural question arises: how does the introduction of fractional operators influence General Relativity, Einstein field equations, black hole solutions and cosmological models, particularly in addressing key challenges within the current cosmological framework.

In this study, we aim to explore the consequences of fractional gravity to General Relativity and the field of Cosmology and whether it can provide a competitive description with the added non-locality the fractional operator inherits. We analyze whether this theory can potentially address the current challenges of the standard cosmological model while maintaining consistency with it at cosmological-scale observations.

There are two primary ways to incorporate fractional operators into a gravitational theory. The most common approach, known as Last Step Modification, applies fractional field equations after deriving the cosmological field equations, offering a non-fundamental but simpler methodology. The second approach, First Step Modification, introduces fractional derivatives from the outset, establishing a fractional geometric framework. Calcagni's renormalizable theories fall in the latter category and thus the difficulty in finding exact solutions [1]. It involves generalizing all derivatives of integer order to derivatives of arbitrary (real or complex) order, when defining the Christoffel symbols [4].

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\delta}(D_\nu g_{\delta\lambda} + D_\lambda g_{\delta\nu} - D_\delta g_{\nu\lambda}), \tag{1.1}$$

where $\mu, \nu = 0, 1, 2, 3$ are space-time indices and D_ν stands for the fractional derivative with respect to the coordinate ν which in the classical limit coincides with the usual derivative operator, namely $D_\nu \to \partial_\nu$ (details about this limit can be found in the next chapter). Note that if we define the fractional covariant derivative $\tilde{\nabla}$ of a (k, l)-rank tensor T as

$$\begin{aligned}\tilde{\nabla}_\sigma T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} = D_\sigma T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\ - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\lambda\cdots\nu_l} - \cdots\end{aligned} \tag{1.2}$$

it is compatible with the metric in the sense that [4]

$$\tilde{\nabla}_\mu g_{\nu\rho} = 0. \tag{1.3}$$

The Riemann tensor, Ricci tensor, and Ricci scalar can now be defined as [4]

$$\begin{aligned} R_{\rho\mu\sigma\nu} &= D_\sigma \Gamma^\alpha_{\mu\nu} - D_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\tau_{\mu\nu}\Gamma^\alpha_{\sigma\tau} - \Gamma^\tau_{\mu\sigma}\Gamma^\alpha_{\nu\tau} && (1.4)\\ R_{\mu\nu} &= D_\sigma \Gamma^\sigma_{\mu\nu} - D_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\tau_{\mu\nu}\Gamma^\sigma_{\sigma\tau} - \Gamma^\tau_{\mu\sigma}\Gamma^\sigma_{\nu\tau} && (1.5)\\ R &= g^{\mu\nu} R_{\mu\nu}. && (1.6) \end{aligned}$$

It is worth emphasizing that Eqs. (1.1)-(1.6) correspond to straightforward generalizations because we only replaced the usual derivative operator with the fractional one. Indeed, these expressions serve as a good starting point because, as mentioned earlier, we are assuming that the "standard" geometry only emerges in the limit where D_μ coincides with ∂_μ. Of course, it would be interesting to start from first principles, such as considering, the definition of the Riemann tensor

$$R(X,Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \nabla_{[X,Y]} \tag{1.7}$$

where $[X,Y]$ is the Lie bracket of vector fields and $[\nabla_X, \nabla_Y]$ is the commutator of differential operators. However, this raises other questions about what fractional Lie brackets are or how to generalize the commutator between vector fields in such a way that yields (1.4). Another question is whether these commutators obey a certain fractional Jacobi identity that leads to the Bianchi identity, namely

$$\tilde{\nabla}_{[\alpha} R^\lambda_{\beta\gamma]\delta} = 0. \tag{1.8}$$

Answering these questions is not trivial, given the lack of the usual Leibniz rule for fractional operators. Indeed, a Leibniz rule exists for fractional derivatives, but it contains an infinite series of classical derivatives of the functions involved, making it difficult to handle [5–7]. The intention of this work is not to answer these questions but rather to leave open the possibility of exploring these details in future work.

Now, by considering (1.1)-(1.6) as the starting point, we have two routes to obtain Einstein's field equations: i) solving the variational problem from the Einstein-Hilbert action or ii) considering the Einstein field equations as a starting point by using (1.5) and (1.6). If we start from the Einstein-Hilbert action, we encounter, after variations with respect to the metric, the boundary term

$$\delta g^{\mu\nu}\mathcal{O}_{\mu\nu} = \nabla_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\rho_{\mu\rho}) \tag{1.9}$$

that vanishes in the case of ordinary derivatives but generally does not vanish when fractional operators are used (see [4] for details). By contrast, if we consider the Einstein field equations as fundamentals, the boundary term is (apparently) absent, and we have

$$R_{\mu\nu} - \frac{1}{2}R = \kappa^2 T_{\mu\nu}. \tag{1.10}$$

Note that if (1.8) is true, is straightforward to show that $\tilde{\nabla}_\mu(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 0$, from which $\tilde{\nabla}_\mu T^{\mu\nu} = 0$ which means that the energy-momentum tensor in (1.10) is conserved in the fractional sense. Regardless of the case, in this work, we will consider (1.10) as true meaning that, in order to be compatible with the equations obtained from the Einstein-Hilbert action, the energy-momentum tensor must be conceived as a quantity that contains the matter sector and the corrections introduced by $\mathcal{O}_{\mu\nu}$, that is,

$$T_{\mu\nu} \rightarrow \; T_{\mu\nu} + \mathcal{O}_{\mu\nu}. \tag{1.11}$$

This work is organized as follows. The next chapter 2, introduces the formal aspects of the new fractional derivative. Next, in chapter 3, we deduce the set of Einstein field fractional equations for the BTZ black hole. In chapter 4 we deduce the equations for Friedmann type spacetime Cosmology. In particular, we explore the consequences of assuming material and radiation dominance in flat metric $k = 0$ as a solution to the fractional equations. Finally, the last chapter is devoted to some final comments and remarks.

Capítulo 2

Fractional Calculus

In this chapter, we present the basic definition and notation related to fractional calculus used in this study. As we aim for this work to be as self-contained as possible, we will start with the definitions of the Caputo and Riemann-Liouville derivatives to lay the groundwork for our new fractional derivative.

Let us start with the definition of the Riemann integral which plays a central role in the definition of the fractional derivatives used here.

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha > 0$ is given by (see [5–7])

$$\left(I_{a^+}^{\alpha} h\right)(x) \;= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{h(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > a. \tag{2.1}$$

We denote by $I_{a^+}^{\alpha}(L_1)$ the class of functions h, represented by the fractional integral (2.1) of a summable function, that is $h = I_{a^+}^{\alpha}\varphi$, where $\varphi \in L_1(a,b)$. A description of this class of functions was provided in [5, 7].

Theorem 2.2. *A function $h \in I^{\alpha}_{a^+}(L_1), \alpha > 0$, if and only if its fractional integral $I^{s-\alpha}_{a^+}h \in AC^s([a,b])$, where $s = [\alpha]+1$ and $(I^{s-\alpha}_{a^+}h)^{(k)}(a) = 0$, for $k = 0, \dots, s-1$.*

In Theorem 2.2, $AC^s([a,b])$ denotes the class of functions h, which are continuously differentiable on the segment $[a,b]$, up to order $s-1$ and $h^{(s-1)}$ is absolutely continuous on $[a,b]$. By removing the last condition in Theorem 2.2, we obtain a class of functions that admit a summable fractional derivative. (See [5, 7])

Definition 2.3 (see [7])**.** A function $h \in L_1(a,b)$ has a summable fractional derivative $\left(D^{\alpha}_{a^+}h\right)(x)$ if

$$\left(I^{s-\alpha}_{a^+}h\right)(x) \in AC^s([a,b]),$$

where $s = [\alpha] + 1$.

2.1. Riemann-Liouville fractional derivative

Definition 2.4. Let $\left(D^{\alpha}_{a^+}h\right)(x)$ denote the **fractional Riemann–Liouville derivative** of order $\alpha > 0$ (see [5–7])

$$\begin{aligned} \left(D^{\alpha}_{a^+}h\right)(x) &= \left(\frac{d}{dx}\right)^s \frac{1}{\Gamma(s-\alpha)} \int_a^x \frac{h(t)}{(x-t)^{\alpha-s+1}}\, dt \\ &= \left(\frac{d}{dx}\right)^s \left(I^{s-\alpha}_{a^+}h\right)(x), \end{aligned} \tag{2.2}$$

where $s = [\alpha] + 1, x > a$ $[\alpha]$ denotes the integer part of α and Γ is the gamma function.

When $0 < \alpha < 1$, then (2.2) takes the form

$$\left(D_{a^+}^{\alpha} h\right)(x) \; = \frac{d}{dx}\left(I_{a^+}^{1-\alpha} h\right)(x). \tag{2.3}$$

Note that, when $\alpha \to 1$, we recover the typical derivative operator [5–7].

The semigroup property for the composition of fractional derivatives does not generally hold (see [6, Sect. 2.3.6]). In fact, the property:

$$D_{a^+}^{\alpha}\left(D_{a^+}^{\beta} h\right) = D_{a^+}^{\alpha+\beta} h, \tag{2.4}$$

holds whenever

$$h^{(j)}(a^+) = 0, \qquad j = 0, 1, \ldots, s-1, \tag{2.5}$$

and $h \in AC^{s-1}([a, b])$, $h^{(s)} \in L_1(a, b)$ and $s = [\beta] + 1$. Thus, we can write this result in the following:

Lemma 2.5. *Consider $h \in AC^{s-1}([a, b])$ and $h^{(s)} \in L_1(a, b)$ then,*

$$D_{a+}^{\alpha}\left(D_{a+}^{\beta}\right) h = D_{a+}^{\beta}\left(D_{a+}^{\alpha}\right) h, \tag{2.6}$$

holds whenever

$$h^{(j)}(a^+) = 0, \quad j = 0, 1, \ldots, s-1, \tag{2.7}$$

where $s = [\beta] + 1$.

Demostración. This proof can be found in [6, Section 2.3.6]. $\square$

Remark 2.6. It is worth noticing that the **Riemann-Liouville derivative of a constant is not zero**. However, in the limit process, it behaves as expected.

$$\lim_{\alpha \to 1} \left(D^{\alpha}_{a^+} 1 \right)(x) = \lim_{\alpha \to 1} \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} = 0. \tag{2.8}$$

2.2. Caputo fractional derivative

Definition 2.7. Let $\alpha \geq 0$ and $m = [\alpha]$. We can then define **the Caputo derivative** [5–7] ${}_cD^{\alpha}_{a^+}$ as

$$ {}_cD^{\alpha}_{a^+} f = I^{m-\alpha}_{a^+} \left(\frac{d}{dx} \right)^m f, \tag{2.9}$$

when $\left(\frac{d}{dx} \right)^m f \in L_1[a, b]$.

Note that Caputo derivative of a constant is zero, as expected. This presents an advantage when constructing Christoffel symbols because, for example, if the metric depends only on the radial coordinate, its derivative with respect to time vanishes (something that would not occur when using the Riemann-Liouville derivative). However, the use of Caputo is problematic when constructing the term

$$ {}_cD^{\beta}_{0^+} (r^{-2} {}_cD^{\alpha}_{0^+} r^2) = -\frac{2\alpha}{\Gamma(1-\alpha)\Gamma(3-\alpha)} \int_0^r \frac{t^{-\alpha-1}}{(r-t)^{\beta}} dt \tag{2.10}$$

because it fails to converge when $r \to 0$ and $\alpha > 0$. One option is to contemplate the

Riemann integral from a certain point $a > 0$. As we will see later, this term arises naturally from Christoffel symbols for metrics that are circularly and spherically symmetric. However, this approach introduces additional challenges because the solution becomes approximate rather than exact, requiring numerical computation for a given fixed a. Moreover, when employing standard derivatives, this issue is absent, and our aim here is to ensure that our equations inherit all the desirable properties. Now, although the Riemann-Liouville derivative does not diverge in this case, it is not useful for deriving the Einstein equations, because, as we saw earlier, the derivative of a constant is not zero. For this reason, we propose a modification to the Riemann-Liouville operator such that its action on a constant vanishes, as we see in what follows.

2.3. Modified fractional derivative

Definition 2.8. Consider $q_1(x,\alpha)$ a continuous function, $q_2(x,\alpha)$ a continuously differentiable function on x and let $\left({}^{(q_1,q_2)}D^{\alpha}_{a^+}h\right)(x) = \left({}^{\overline{q}}D^{\alpha}_{a^+}h\right)(x)$ denote the **$\overline{q}$-weighted fractional Riemann-Liouville derivative** of order $\alpha > 0$. For $q_1, q_2 \in AC^s(\mathbb{R})$

$$\left({}^{\overline{q}}D^{\alpha}_{a^+}h\right)(x) = q_1(x,\alpha)\left(\frac{d}{dx}\right)^s q_2(x,\alpha)\left(I^{s-\alpha}_{a^+}h\right)(x), \tag{2.11}$$

where $s = [\alpha]+1, x > a$ and $[\alpha]$ denotes the integer part of α with $0 < \alpha < 1$ and $\lim\limits_{\alpha\to 1} q_1(x,\alpha) = \lim\limits_{\alpha\to 1} q_2(x,\alpha) = 1$. As we will see later, for convenience we take

$q_2(x,\alpha) = (x-a)^{\alpha-1}$ so then (2.11) takes the form

$$\left({}^{\overline{q}}D^{\alpha}_{a^+}h\right)(x) = q_1(x,\alpha)\left(\frac{d}{dx}\right)(x-a)^{\alpha-1}\left(I^{1-\alpha}_{a^+}h\right)(x). \tag{2.12}$$

It is not diffficult to see that the operator is linear; however, the central motivation for considering this operator is that, although it has a structure similar to that of the Riemann-Liouville operator, the derivative of a constant is zero

$$\left({}^{\overline{q}}D^{\alpha}_{a^+}1\right)(x) = 0 \tag{2.13}$$

Another consequence arises when we differentiate polynomials. If we consider $a = 0$, $P(x) = x^n$ and $x, n > 0$,

$$\left({}^{\overline{q}}D^{\alpha}_{0^+}P(x)\right)(x) = \frac{nq_1(x,\alpha)x^{n-1}\Gamma(n+1)}{\Gamma(n-\alpha+2)}, \tag{2.14}$$

and $\lim\limits_{\alpha\to 1}\left({}^{\overline{q}}D^{\alpha}_{0^+}P(x)\right)(x) = nx^{n-1}$.

As in the case of the Riemann-Liouville operator, the semigroup property for the composition of fractional derivatives is generally not satisfied; however, the following lemma is useful

Lemma 2.9. *Consider* $q_2(x,\alpha) \in AC^s([a,b]\times(0,1))$, $h \in AC^{s-1}([a,b])$ *and* $h^{(s)} \in L_1(a,b)$ *then,*

$${}^{\overline{q}}D^{\alpha}_{a^+}\left({}^{\overline{q}}D^{\beta}_{a^+}\right)h = {}^{\overline{q}}D^{\beta}_{a^+}\left({}^{\overline{q}}D^{\alpha}_{a^+}\right)h, \tag{2.15}$$

holds whenever

$$h^{(j)}(a^+) = 0, \quad j = 0, 1, \ldots, s-1, \tag{2.16}$$

where $s = [\beta] + 1$.

Demostración. The proof can be followed using the formula

$$\left({}^{\overline{q}}D^{\alpha}_{a^+}h\right)(x) = q_1(x,\alpha)\Big[\left(\frac{d}{dx}\right)^s q_2(x,\alpha)\Big]\left(I^{s-\alpha}_{a^+}h\right)(x) + q_2(x,\alpha)\left(D^{\alpha}_{a^+}h\right)(x) \tag{2.17}$$

and the result of lemma 2.5 □

Note that, up to this point, the weight q_2 is general. However, to avoid singularities in (2.10), it is convenient to consider

$$q_2(x,\alpha) = \frac{1}{q_1(x,\alpha)} = (x-a)^{1-\alpha}. \tag{2.18}$$

Indeed, in this case we obtain

$$D^{\beta}_{0^+}(r^{-2}D^{\alpha}_{0^+}r^2) = -\frac{4r^{-\beta-\alpha}\beta\Gamma(1-\alpha)}{\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)} \tag{2.19}$$

At this point, some comments are in order. First, note that we have used different fractional indices that translates to using α to define the Christoffel symbols in (1.1) and β to define the curvature tensor (1.4). Indeed, this is the most general way to consider fractional Einstein equations. Furthermore, if they are considered equal a priori, the result of (2.19) does not lead to the result obtained with the ordinary derivative in the limit. Second, the result obtained with the classical derivative is achieved if $\beta \to 1$ is taken before $\alpha \to 1$, otherwise, the expression diverges. Of course, this introduces a hierarchy in how the classical limit is recovered: first turning off non-locality in curvature, and then in the Christoffel symbols.

In the next section, we derive the Einstein field equation for a static and circularly symmetric metric by assuming the weights in (2.18) and taking $a = 0$. To make the equations readable, we use the notation

$$({}^{\overline{q}}D^{\alpha}_{0+})_r \equiv d^{\alpha}. \tag{2.20}$$

Capítulo 3

The Banados-Teitelboim-Zanelli Solution

In this section we seek solutions to the field equations, which in this case translates to dealing with a system of 10 integro-differential equations as each fractional derivative operator involves an integral, as shown in the previous chapter. The standard approach to reducing the number of equations to be solved is to assume a system with certain symmetries and a particular parameterization for the metric. For instance, in $3+1$ spacetime dimensions, the simplest assumption is a spherically symmetric and static system, whose line element is given by

$$ds^2 = -F(r)dt^2 + G(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.1}$$

As is known (and expected), Einstein equations depend only on the radial coordinate. However, it is worth mentioning that some non-zero Christoffel symbols explicitly contain the coordinate θ, which does not appear in the final result owing to Leibniz's rule and trigonometric identities. However, as stated above, in fractional calculus there is no standard Leibniz's rule but a relation involving an infinite series in derivatives which is not trivial to deal with. In this regard, we have to either find the convergence of the series (which is not trivial) or impose constraints to ensure the classical limits. For example, in the case of (3.1) we encounter the terms

$$\begin{aligned} D_3\left(\frac{D_3\sin^2\theta}{\sin^2 2\theta}\right)+\frac{D_3(D_3\sin^2\theta)}{\sin^2\theta} &= -2 \qquad (3.2)\\ \left(\frac{D_3\sin^2\theta}{\sin^2\theta}\right)^2 - D_3\left(\frac{D_3\sin^2\theta}{\sin^2 2\theta}\right)+\frac{D_3(D_3\sin^2\theta)}{\sin^2\theta} &= 0, \qquad (3.3)\end{aligned}$$

where D_3 is the fractional derivative with respect to θ, that come from the computation of the G_{22} and G_{33} of the Einstein tensor (note that, if we assume $D_3 = \partial_3 = \partial/\partial\theta$ the constraints are automatically satisfied). One way to avoid this "polar angle problem" is by studying a toy model such as Einstein's equations in $2+1$ dimensional spacetime where for circularly symmetric and static situations the metric reads

$$ds^2 = -F(r)dt^2 + G(r)dr^2 + r^2 d\phi^2 \qquad (3.4)$$

There has been a significant amount of research focused on the study of three-dimensional gravity [8–10]. These models provide valuable insights into their higher-dimensional analogs, specially when it comes to exploring concepts or techniques that are directly intractable in four or more dimensions, providing a testing ground for techniques such as holography, perturbation methods, and the renormalization of gravitational theories. Furthermore, models in three dimensions, such as the BTZ black hole solution, have played a crucial role in exploring thermodynamic properties, quantum aspects of black holes and the holographic framework [11–13], bridging insights to higher-dimensional settings.

3.1. Fractional Einstein Equations

In what follows, we obtain the explicit form of the fractional Einstein field equations for the $2+1$ dimensional metric

$$ds^2 = -F(r)dt^2 + G(r)dr^2 + R(r)d\phi^2, \tag{3.5}$$

from (1.1), (1.5), (1.6), and (1.10), with the convention that the index α is used in the definition of the Christoffel symbols and β in the definition of the Riemann tensor. Note that, because the metric is a function of the radial coordinate, only terms involving the fractional derivative with respect to the radius will persist.

Using (2.20), the fractional Einstein field equations can be written as

$$\begin{aligned}
-T_0^0 &= -\frac{(d^\alpha F)^2}{32\pi F^2 G} + \frac{d^\alpha F d^\alpha G}{32\pi F G^2} + \frac{d^\beta\left(G^{-1}d^\alpha F\right)}{32\pi F} - \frac{d^\beta\left(F^{-1}d^\alpha F\right)}{32\pi G} \\
&\quad -\frac{d^\beta\left(R^{-1}d^\alpha R\right)}{32\pi G} - \frac{d^\beta\left(G^{-1}d^\alpha R\right)}{32\pi R} \qquad (3.6)\\
T_1^1 &= -\frac{(d^\alpha F)^2}{32\pi F^2 G} + \frac{d^\alpha F d^\alpha G}{32\pi F G^2} + \frac{d^\beta\left(G^{-1}d^\alpha F\right)}{32\pi F} - \frac{d^\beta\left(F^{-1}d^\alpha F\right)}{32\pi G} + \frac{d^\alpha F d^\alpha R}{32\pi F G R} \\
&\quad +\frac{d^\alpha G d^\alpha R}{32\pi G^2 R} - \frac{(d^\alpha R)^2}{32\pi G R^2} + \frac{d^\beta\left(G^{-1}d^\alpha R\right)}{32\pi R} - \frac{d^\beta\left(R^{-1}d^\alpha R\right)}{32\pi G} \qquad (3.7)\\
T_2^2 &= \frac{d^\beta\left(F^{-1}d^\alpha F\right)}{32\pi G} + \frac{d^\beta\left(G^{-1}d^\alpha F\right)}{32\pi F} - \frac{d^\alpha G d^\alpha R}{32\pi G^2 R} + \frac{(d^\alpha R)^2}{32\pi G R^2} \\
&\quad +\frac{d^\beta\left(R^{-1}d^\alpha R\right)}{32\pi G} - \frac{d^\beta\left(G^{-1}d^\alpha R\right)}{32\pi R}. \qquad (3.8)
\end{aligned}$$

It is worth noticing that the dimensions of the field equations change depending on the choice of weights (q_1, q_2). However, with the choice in equation (2.18), the dimensions of the fractional derivative of the metric are length$^{-\alpha}$. Consequently, the dimensions of the Einstein equations are length$^{-2\alpha}$ and length$^{-\alpha-\beta}$ so thus, when $\alpha \to 1$ and $\beta \to 1$ we obtain the correct dimensions (in natural units). From a physical standpoint, this means that we must be careful when assigning the meaning of each component of the fractional Einstein tensor $G_{\mu\nu}$ to match the energy-momentum tensor $T_{\mu\nu}$. More precisely, in the standard case, we assign $T^\mu_\nu = (\rho, -p_r, -p_t)$ with ρ, p_r and p_t as the energy density, radial pressure and tangential pressure, respectively, and each quantity has dimensions of length^{-2} but when applying fractional operators it is convenient to redefine

$$d^\alpha \to \Xi^{\alpha-1} d^\alpha \qquad (3.9)$$

$$d^\beta \to \Xi^{\beta-1} d^\beta \qquad (3.10)$$

with Ξ, some quantity with dimensions of length (which can be taken as a constant associated with some characteristic length of the model under study) so that $\{\rho, p_r, p_t\}$ have the correct dimensions.

The next step in the program is to solve the set of fractional derivatives (3.6)-(3.8) with $R = r^2$ so we must solve the problem of solving three integro-differential equations with five unknowns, namely, $\{\rho, p_r, p_t, F, G\}$ which represents a non-trivial challenge. We can try by following the routes we explore in standard General Relativity, namely

1. Provide an equation of state relating ρ and p_r and a suitable anisotropic function.
2. Consider a matter sector based on fundamental fields.
3. Provide some geometric restrictions
4. Consider a vacuum solution.

3.1.1. Fractional BTZ Black Hole

The standard Einstein field equations in $2 + 1$ dimensions admit a black hole solution with negative cosmological constant [14, 15]. It is the only non-trivial vacuum solution since in $2 + 1$ dimensions the metric is completely determined by the mass-energy distribution and without a cosmological constant ($\Lambda = 0$) the solution is locally flat.

The solution was originally derived from the Einstein-Hilbert action [14]

$$S = \int \sqrt{-g}(R - 2\Lambda)d^2xdt \tag{3.11}$$

with negative cosmological constant ($\Lambda = -1/\ell^2$) and metric.

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2\left(N^\phi dt + d\phi\right)^2 \tag{3.12}$$

with lapse $N(r)$ and angular shift $N^\phi(r)$

$$N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \quad N^\phi(r) = -\frac{J}{2r^2} \tag{3.13}$$

and constants of integration M and J appearing and corresponding to mass and angular momentum, respectively. This solution serves as a natural analog to the classical black hole to study key features without the unnecessary complications and more manageable equations. As discussed previously, the BTZ model has been extensively inspiring studies in quantum gravity, black hole thermodynamics, and holography, owing to its simplicity and theoretical richness.

The metric further simplifies considering a non-rotating black hole $J = 0$, which in the static and circularly symmetric regime, reads

$$ds^2 = -\left(-M + \frac{r^2}{\ell^2}\right)dt^2 + \frac{1}{\left(-M + \frac{r^2}{\ell^2}\right)}dr^2 + r^2d\phi^2, \tag{3.14}$$

which corresponds to the static BTZ black hole with event horizon $r_H = \ell\sqrt{M}$. Alternatively, the BTZ black hole can be thought of as a solution of Einstein's equation without cosmological constant but supported by a matter sector given by $T^{\mu}_{\nu} = \frac{1}{8\pi\ell^2} diag(1,1,1)$. In this work, instead of trying to solve the set (3.6)-(3.8), we assume (3.14) as a solution to the fractional equations and explore the behavior of the corresponding fractional matter sector. Before proceeding with the calculation, we would like to point out that the choice of a metric that solves the classical (non-fractional) Einstein equations is made for simplicity. One could attempt to use a metric that depends on the fractional parameters and coincides with BTZ in the appropriate limit. However, although we explored this approach, we did not find a metric simpler than the BTZ.

After using (3.14) in the fractional equations, we obtain expressions for the matter sector that are not included here due to their length, as they involve hypergeometric functions and are not particularly illuminating. Furthermore, since our focus is on the behavior near the realm of General Relativity, we consider the solution's behavior for α, β close to one. By expanding in series for these parameters, we find that for fractional parameters approaching one, the standard notions of space-time as described by General Relativity are recovered. This approach allows us to understand how deviations from classical general relativity manifest while emphasizing the consistency with the well-established classic theory

$$-T^0_0 = \frac{1}{8\pi\ell^2}\left(1+\frac{\beta-1}{2(\alpha-1)}\right)+\frac{M(\beta-1)}{16\pi r^2(\alpha-1)}+O\left(\alpha-1,\beta-1\right) \tag{3.15}$$

$$T^1_1 = \frac{1}{8\pi\ell^2}\left(1-\frac{\beta-1}{2(\alpha-1)}\right)-\frac{M(\beta-1)}{16\pi r^2(\alpha-1)}+O\left(\alpha-1,\beta-1\right) \tag{3.16}$$

$$T^2_2 = \frac{1}{8\pi\ell^2}\left(1+\frac{\beta-1}{2(\alpha-1)}\right)+\frac{M(\beta-1)}{16\pi r^2(\alpha-1)}+O\left(\alpha-1,\beta-1\right). \tag{3.17}$$

At this point some comments are in order. First, note that expressions (3.15-3.17) underscore the critical requirement of preserving hierarchy in the fractional parameters when taking the limit to recover the classic solution. Notably, it is essential that β approaches 1 prior to α, as β must reach this value first to ensure the correct limiting process. Second, around this limit, the introduction of non-locality leads to a small correction in the cosmological constant. In fact, the asymptotic behavior corresponds to a solution with an anisotropic cosmological constant. Finally, the second term in the matter sector $\propto r^{-2}$ is reminiscent of the charged BTZ black hole solution with line element [16]

$$F = \frac{1}{G} = -M + \frac{r^2}{\ell^2} - \frac{Q^2}{2} \ln \frac{r}{r_0}, \tag{3.18}$$

where Q represents the electric charge of the black hole and r_0 is an arbitrary reference scale. Solving the Einstein Field Equations for this metric the matter sector is

$$\begin{aligned} -T_0^0 &= \frac{1}{8\pi\ell^2} + \frac{Q^2}{32\pi r^2} && (3.19) \\ T_1^1 &= \frac{1}{8\pi\ell^2} - \frac{Q^2}{32\pi r^2} && (3.20) \\ T_2^2 &= \frac{1}{8\pi\ell^2} + \frac{Q^2}{32\pi r^2}. && (3.21) \end{aligned}$$

Upon comparing (3.15-3.17) with equations (3.19-3.21), we establish the identification $Q^2 = 2M(\beta - 1)/(\alpha - 1)$. We can conclude, then, that the non-locality introduced by the fractional derivative leaves traces in the classical results, providing anisotropy to the cosmological constant and an effective electric charge. This result resembles, to some extent, the Kaluza-Klein mechanism (see [17], for example), where a gravitational theory in 5-dimensional spacetime leads to a theory of gravity coupled with electrodynamics in 4-dimensional spacetime after compactification. In this case, compactification is replaced by non-locality. We want to emphasize that the mechanism described here is unrelated to Kaluza-Klein, but the resemblance is interesting.

At this stage, after analyzing our initial fractional solution, we observe that certain simplifications typical of classical physics, such as the absence of radial dependence, do not occur in the fractional case. This prompts us to consider whether the observed non-homogeneity arises from the application of fractional derivatives to the r^2 scaling of angular distances. However, implementing an isotropic coordinate system, as is done for the Schwarzschild black hole [18, 19], results in metrics that are analytically intractable when approached via fractional integrals. In contrast, cosmology, specially cosmology in flat spatial geometry, benefits from the simplicity of the spatial part $dx^2 + dy^2 + dz^2$ being Euclidean, and the complexity of the universe's expansion is entirely captured by the time-dependent scale factor.

Capítulo 4

Cosmology

Observational data from astrophysical and cosmological probes has established that baryonic matter constitutes only about 15 % of the total matter content of the universe, with the remainder attributed to a mysterious 'dark matter' (DM) component. Some evidence include the kinematics of spiral galaxies [20–22], cosmic microwave background and large-scale structure observations [23, 24], among many others. However, despite extensive efforts [25, 26] no direct detection of dark matter particles has been made. This has spurred the exploration of alternative explanations, including modifications to General Relativity that do not rely solely on standard particle candidates [27, 28].

Modified gravity theories have emerged as promising alternatives to explain the late-time acceleration of the universe [29–32], as well as to try to alleviate the H_0 tension and the cosmological constant 'dark energy' (DE) problem to better align with observational data. Fractional calculus has shown potential in these areas by modifying the Friedmann equations, describing inflationary cosmologies in both FLRW and Bianchi metrics [32, 33], and even proposing the replacement of the cosmological constant with fractional dissipative forces [31, 34, 35]. Recent work has also explored the possibility that the elusive dark matter component is replaced by the mathematical structure of fractional gravity [36, 37], suggesting a deeper connection between non-locality and the fundamental properties of the universe. This is mainly because, in fractional Cosmology, standard evolution of the cosmic species densities has been found to depend on the fractional parameters of the theory [31, 33, 36, 37].

This study sets itself apart from prior research in fractional cosmology, which have primarily centered around last-step modifications and fractional effective actions. In contrast, our study emphasize first-step modifications directly to EFE, which, to our knowledge, has been largely confined to the Minkowski universe due to the mathematical complexity of fractional differential operators. By applying the fractional framework and leveraging its inherent non-locality coherently at a foundational level to the Einstein Field Equations, this approach has the potential to yield novel insights or results, offering a fresh perspective on gravitational theory and cosmological modeling.

4.1. Solutions in $2+1$ dimensional Cosmology

Cosmological models have also been generalized to lower-dimensional spacetime [38–41] in an attempt to answer Universe's large-scale homogeneity problem.

The presence of horizons around any comoving observer determines a maximum possible coherence radius over which one might expect homogeneity [41]. All standard Friedmann models predict the existence of horizons, except in the case of an empty Universe. However, it has been shown that models of isotropic, matter-dominated universes in 2+1 dimensions do not exhibit horizons [41]. This has consequently increased interest in exploring models that extend beyond the standard cosmological framework. More recently, these lower-dimensional cosmological models have been utilized to explore the cosmic holographic principle [42, 43].

The 2+1 Friedmann type (spatially homogeneous and isotropic) cosmological model line element is [40]

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right) \tag{4.1}$$

where $a(t)$ is the scale factor and $k = -1, 0, 1$ for hyperbolic, flat and circular two-dimensional spatial geometry, respectively. The corresponding Einstein field equations are [9]

$$2\pi\rho = \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \tag{4.2}$$

$$2\pi p = -\frac{\ddot{a}}{a}, \tag{4.3}$$

where ρ is the energy density and p the pressure of the fluid. The corresponding energy conservation equation is [9]

$$\frac{d}{dt}\left(\rho a^2\right) + p\frac{d}{dt}\left(a^2\right) = 0. \tag{4.4}$$

A matter-dominated Universe follows the dust EoS $p = 0$. Thus, the energy conservation equation states $\rho a^2 = \text{const}$. In standard Friedmann fashion the scale factor can now be derived from the first field equation (4.2).

$$a(t) = a_0 \pm \sqrt{2M - k}\,(t - t_0) \tag{4.5}$$

where $M = \pi \rho_0 a_0^2$. For a radiation-dominated Universe $p = \frac{1}{2}\rho$. According to (4.4) $\rho a^3 = \text{const}$. The complete scale factor solution can be found in [9]. For early times $a \propto t^{2/3}$.

In what follows, we obtain the explicit form of the fractional Einstein field equations for the $2+1$ dimension cosmological metric. For simplicity, consider the Cartesian flat metric ($k = 0$)

$$ds^2 = dt^2 - A(t)\left(dx^2 + dy^2\right) \tag{4.6}$$

where $A(t) = a(t)^2$ is the squared scale factor. The scale factor is hidden in A so that the classic Leibniz rule is not unintentionally applied in the context of fractional calculus. The advantage of using Cartesian coordinates is that only terms involving the fractional derivative with respect to time will persist. This is not true for the standard metric (4.1) where additional radial fractional derivatives appear, their effects are discussed in appendix A. The convention is that the index α is used in the definition of the Christoffel symbols. Using the notation (2.20)

$$\begin{aligned} \Gamma^0_{11} &= \Gamma^0_{22} = \frac{d^\alpha A}{2} \\ \Gamma^1_{01} &= \Gamma^2_{02} = \frac{d^\alpha A}{2A}. \end{aligned} \tag{4.7}$$

While a different index, β is used in further derivation in the definition of the Riemann tensor

$$\begin{aligned}
R^0_{110} = R^0_{220} &= -\frac{d^\beta(d^\alpha A)}{2} + \frac{(d^\alpha A)^2}{4A} \\
R^1_{010} = R^2_{020} &= -\frac{d^\beta\left(A^{-1}d^\alpha A\right)}{2} - \frac{(d^\alpha A)^2}{4A^2} \\
R^1_{221} = -R^2_{121} &= -\frac{(d^\alpha A)^2}{4A},
\end{aligned} \tag{4.8}$$

where $\alpha < \beta < 1$ as seen in (2.19). With these considerations, the fractional Einstein field equations are succinctly expressed

$$T^0_0 = -\frac{(d^\alpha A)^2}{8\pi A^2} - \frac{d^\beta\left(A^{-1}d^\alpha A\right)}{4\pi} + \frac{d^\beta\left(d^\alpha A\right)}{4\pi A} \tag{4.9}$$

$$T^1_1 = T^2_2 = \frac{(d^\alpha A)^2}{8\pi A^2} + \frac{d^\beta\left(A^{-1}d^\alpha A\right)}{4\pi}. \tag{4.10}$$

The next step in the program is to solve equations (4.9) and (4.10). We will not derive the scale factor as in standard 2+1 Cosmology as it represents a non-trivial challenge of integro-differential equations. Instead the scale factor for matter and radiation-dominated eras are considered in order to analyze the effects of non-locality as introduced by the fractional derivative.

4.1.1. Matter-dominated Universe

The scale factor for matter-dominated 2+1 Universe is time proportional $a \propto t$ (4.5). Thus, the squared scale factor is $A \propto t^2$. For simplicity, taking $A = \left(\frac{t}{t_0}\right)^2$, where t_0 is a reference time, and solving (4.9) and (4.10) gives

$$\rho(t) = -\frac{2t^{-2\alpha}}{\pi\Gamma(4-\alpha)^2} + \frac{t^{-\alpha-\beta}\alpha\Gamma(1-\alpha)}{\pi\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)} + \frac{t^{-\alpha-\beta}(\alpha-2)}{\pi(\alpha-3)\Gamma(4-\alpha-\beta)} \tag{4.11}$$

$$p(t) = -\frac{2t^{-2\alpha}}{\pi\Gamma(4-\alpha)^2} + \frac{t^{-\alpha-\beta}\alpha\Gamma(1-\alpha)}{\pi\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)}. \tag{4.12}$$

The application of the new fractional derivative of order α, β results in the expressions (4.11) and (4.12) for energy density and pressure. In the limit $\alpha, \beta \to 1$ the classic energy density and pressure for $2+1$ matter-dominated flat Universe are recovered

$$\lim_{\alpha,\beta\to 1} \rho(t) = \frac{1}{2\pi t^2} \tag{4.13}$$

$$\lim_{\alpha,\beta\to 1} p(t) = 0 \tag{4.14}$$

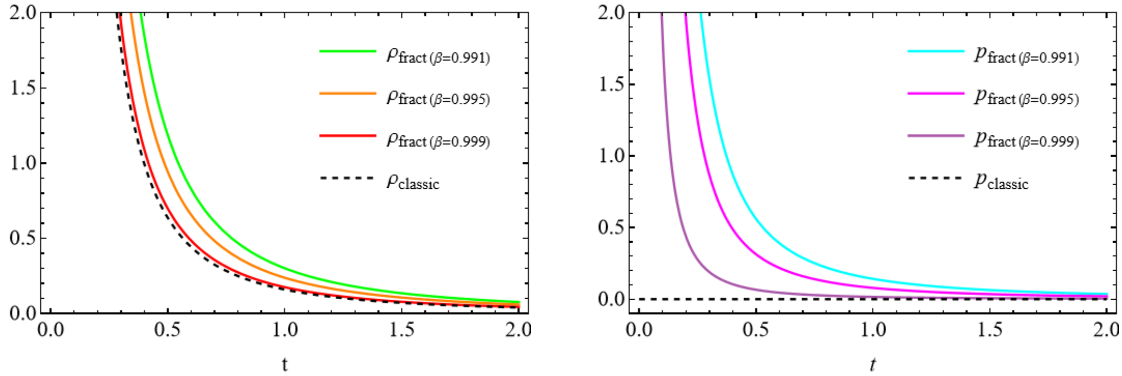
in natural units. By performing a series expansion in (4.11) and (4.12) around $\alpha = 1$ and $\beta = 1$ deeper insight is gained about the implications of the fractional parameters in the solution

$$\rho(t) = \frac{1}{2\pi t^2} + \frac{\beta-1}{2\pi t^2(\alpha-1)} + O\left(\alpha-1, \beta-1\right) \tag{4.15}$$

$$p(t) = \frac{\beta-1}{2\pi t^2(\alpha-1)} + O\left(\alpha-1, \beta-1\right). \tag{4.16}$$

Equations (4.15) and (4.16) emphasize the importance of maintaining the hierarchy among the fractional parameters when taking the limit to retrieve the classical solution with β approaching one first. Around this limit, the introduction of non-locality leads to a small pressure in the dust and a slight modification of the density profile. This effect is reminiscent of the Van der Waals EoS, where corrections are introduced to account for particle interactions and deviations from ideal behavior. Similarly, the fractional derivative introduces subtle corrections that represent minimal internal interactions within the dust, thereby aligning the model more closely with real matter that interacts.

To further elucidate the implications of the fractional derivative within the material sector, the full expressions for $\rho(t)$, $p(t)$ in (4.11) and (4.12) are plotted in Fig.4.1 for $\alpha < \beta < 1$ very close to the classic limit. Notice, as $\beta \to 1$, Fig.4.1a and Fig.4.1b better approximate classic dust density and pressure. Conversely, as $\beta \to \alpha$, the deviation increases since $\alpha \sim \beta$ in (4.15) and (4.16) shifts density and pressure.



(a) Density (b) Pressure

Figura 4.1: Fractional energy density (a) and pressure (b) functions with respect to time for a $2+1$ matter-dominated Universe. α is fixed at 0,99 and β varies.

If non-locality indeed plays a role in shaping the dynamics of the cosmic fluids, then a modification in the Equation of State (EoS) becomes necessary to account for the corrections beyond the perfect fluid approximation. Often the perfect fluids in Cosmology obey the EoS $p = \omega\rho$. For dust, in a matter-dominated Universe

$\omega = 0$. From equations (4.11) and (4.12)

$$\omega(t) = 1 - \frac{t^{-\alpha-\beta}(\alpha-2)}{\rho(t)\pi(\alpha-3)\Gamma(4-\alpha-\beta)}. \tag{4.17}$$

To have, in the limit $\alpha, \beta \to 1$ the classic EoS recovered

$$\lim_{\alpha,\beta\to 1} \omega(t) \;=\; 0. \tag{4.18}$$

Particularly, using the expansions (4.15) and (4.16) around the classic limit

$$\omega = 1 - \frac{1}{1+\left(\frac{\beta-1}{\alpha-1}\right)} + O\left(\alpha-1, \beta-1\right), \tag{4.19}$$

it becomes clear that the relationship between density and pressure is strongly influenced by the fractional parameters and that $\beta \to 1$ prior to α to ensure classic convergence. Moreover, outside the classic limit, there is a small time-dependency in the EoS (4.17).

Dust $(2+1)$	$a(t)$	$2\pi\rho(t)$	$2\pi p(t)$	$\omega(t)$
$(\alpha,\beta) \to (1,1)$	t	$\frac{1}{t^2}$	0	0
$(\alpha,\beta) = (0{,}9, 0{,}91)$	t	$\frac{2{,}56}{t^{1{,}81}}$-$\frac{0{,}83}{t^{1{,}8}}$	$\frac{1{,}61}{t^{1{,}81}}$-$\frac{0{,}83}{t^{1{,}8}}$	$1-\frac{1}{2{,}68-0{,}87t^{0{,}01}}$
$(\alpha,\beta) = (0{,}9, 0{,}95)$	t	$\frac{2{,}23}{t^{1{,}85}}$-$\frac{0{,}83}{t^{1{,}8}}$	$\frac{1{,}25}{t^{1{,}85}}$-$\frac{0{,}83}{t^{1{,}8}}$	$1-\frac{1}{2{,}28-0{,}85t^{0{,}05}}$
$(\alpha,\beta) = (0{,}9, 0{,}99)$	t	$\frac{1{,}9}{t^{1{,}89}}$-$\frac{0{,}83}{t^{1{,}8}}$	$\frac{0{,}9}{t^{1{,}89}}$-$\frac{0{,}83}{t^{1{,}8}}$	$1-\frac{1}{1{,}91-0{,}83t^{0{,}09}}$

Cuadro 4.1: Various fractional expressions for energy density, pressure and dust EoS rounded to two decimal places. In $(\alpha,\beta) \to (1,1)$, $\beta \to 1$ is taken before $\alpha \to 1$

Explicit results for some α, β close to one are presented in the following Tab.4.1, where it is clear that classic zero expressions such as pressure and ω EoS proportionality factor for dust acquire a time dependency. Energy density on the other hand, has its already time dependent expression altered.

4.1.2. Radiation-dominated Universe

The scale factor for a radiation-dominated $2+1$ Universe is $a \propto t^{2/3}$ proportional for early times [9]. Then, the squared scale factor $A \propto t^{4/3}$. For instance, take $A = \left(\frac{t}{t_0}\right)^{4/3}$ where t_0 is a reference time to solve (4.9) and (4.10)

$$\rho(t) = -\frac{2t^{-2\alpha}\Gamma(\frac{7}{3})^2}{9\pi\Gamma(\frac{10}{3}-\alpha)^2} + \frac{t^{-\alpha-\beta}\alpha\Gamma(\frac{7}{3})\Gamma(1-\alpha)}{3\pi\Gamma(\frac{10}{3}-\alpha)\Gamma(2-\alpha-\beta)} + \frac{t^{-\alpha-\beta}(3\alpha-4)\Gamma(\frac{7}{3})}{3\pi(3\alpha-7)\Gamma(\frac{10}{3}-\alpha-\beta)} \tag{4.20}$$

$$p(t) = -\frac{2t^{-2\alpha}\Gamma(\frac{7}{3})^2}{9\pi\Gamma(\frac{10}{3}-\alpha)^2} + \frac{t^{-\alpha-\beta}\alpha\Gamma(\frac{7}{3})\Gamma(1-\alpha)}{3\pi\Gamma(\frac{10}{3}-\alpha)\Gamma(2-\alpha-\beta)} \tag{4.21}$$

Now the application of the fractional derivative results in expressions (4.20) and (4.21) for energy density and pressure. Note that the power laws remain the same as (4.11) and (4.12), only proportionality factors have changed. In the limit $\alpha, \beta \to 1$ the classic energy density and pressure for $2+1$ radiation-dominated Universe are recovered as well.

$$\lim_{\alpha,\beta\to 1} \rho(t) = \frac{2}{9\pi t^2} \tag{4.22}$$

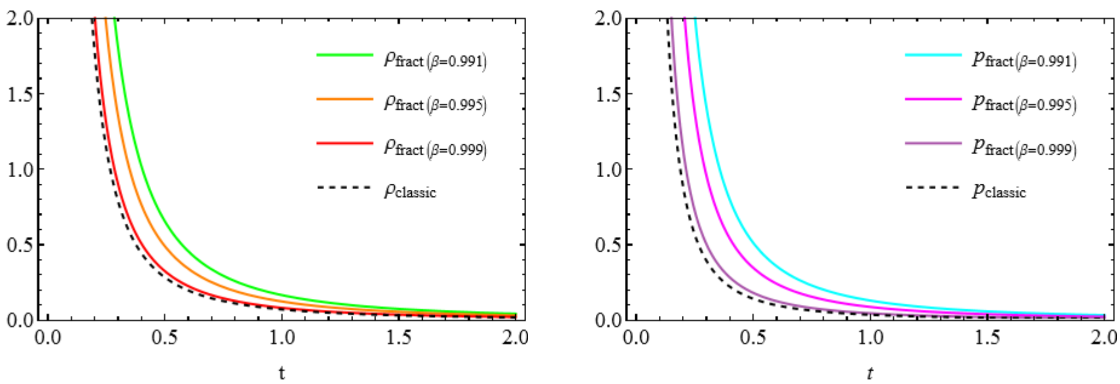
$$\lim_{\alpha,\beta\to 1} p(t) = \frac{1}{9\pi t^2}, \tag{4.23}$$

in natural units. The series expansion in (4.20) and (4.21) around $\alpha = 1$ and $\beta = 1$ yield

$$\rho(t) = \frac{2}{9\pi t^2} + \frac{\beta - 1}{3\pi t^2(\alpha - 1)} + O\left(\alpha - 1, \beta - 1\right) \tag{4.24}$$

$$p(t) = \frac{1}{9\pi t^2} + \frac{\beta - 1}{3\pi t^2(\alpha - 1)} + O\left(\alpha - 1, \beta - 1\right), \tag{4.25}$$

which also suggest a hierarchy in the limits needs to be considered in (4.24) and (4.25) for a convergence to the classic solutions (4.22) and (4.23). As discussed in the previous section, the larger β is with respect to α, the additional term represents a minor correction, which aligns with the behavior under consideration. For radiation expressions $\rho(t), p(t)$ are plotted for certain $\alpha < \beta < 1$ values in Fig.4.2.



(a) Density

(b) Pressure

Figura 4.2: Fractional energy density (a) and pressure (b) functions with respect to time for a $2 + 1$ radiation-dominated Universe. α is fixed at 0,99 and β varies.

In Fig.4.2a and Fig.4.2b fractional energy density and pressure also deviate from the classic expressions as $\beta \to \alpha$. While curves $\beta \to 1$ are closer to the classic density and pressure. Thus, fractional parameters require the same balance discussed in the previous matter-dominate Universe case. Namely, that parameters α, β approach 1 but are separated enough to maintain the hierarchy between them.

Fractional calculus applied to the Einstein field equations during the radiation-dominated era mimics the behavior of the matter-dominated era seen in Fig.4.1. There is an increase in density and pressure values that fades with increasing time, converging toward the classic solution.

In a radiation-dominated Universe $p = \frac{1}{2}\rho$ with $\omega = \frac{1}{2}$. From equations (4.20) and (4.21)

$$\omega(t) = 1 - \frac{t^{-\alpha-\beta}(3\alpha - 4)\Gamma(\frac{7}{3})}{\rho(t)3\pi(3\alpha - 7)\Gamma(\frac{10}{3} - \alpha - \beta)}. \tag{4.26}$$

To have, in the limit $\alpha, \beta \to 1$ the EoS recovered

$$\lim_{\alpha,\beta\to 1} \omega(t) = \frac{1}{2}. \tag{4.27}$$

Specifically, employing the expansions (4.24) and (4.25) in the vicinity of the classical limit,

$$\omega = 1 - \frac{1}{2 + 3\left(\frac{\beta-1}{\alpha-1}\right)} + O\left(\alpha - 1, \beta - 1\right), \tag{4.28}$$

ensuring convergence to the classical behavior requires $\beta \to 1$ before α. Outside this limit $\omega(t)$ is also time-dependent. Like for dust, non-locality acting on radiation seems to slightly alter its behavior over time.

Radiation $(2+1)$	$a(t)$	$2\pi\rho(t)$	$2\pi p(t)$	$\omega(t)$
$(\alpha,\beta)\to(1,1)$	$t^{2/3}$	$\frac{4}{9t^2}$	$\frac{2}{9t^2}$	$\frac{1}{2}$
$(\alpha,\beta)=(0{,}9,0{,}91)$	$t^{2/3}$	$\frac{1{,}37}{t^{1{,}81}}$-$\frac{0{,}39}{t^{1{,}8}}$	$\frac{1{,}1}{t^{1{,}81}}$-$\frac{0{,}39}{t^{1{,}8}}$	$1-\frac{1}{5{,}08-1{,}44t^{0{,}01}}$
$(\alpha,\beta)=(0{,}9,0{,}95)$	$t^{2/3}$	$\frac{1{,}13}{t^{1{,}85}}$-$\frac{0{,}39}{t^{1{,}8}}$	$\frac{0{,}86}{t^{1{,}85}}$-$\frac{0{,}39}{t^{1{,}8}}$	$1-\frac{1}{4{,}18-1{,}44t^{0{,}05}}$
$(\alpha,\beta)=(0{,}9,0{,}99)$	$t^{2/3}$	$\frac{0{,}89}{t^{1{,}89}}$-$\frac{0{,}39}{t^{1{,}8}}$	$\frac{0{,}62}{t^{1{,}89}}$-$\frac{0{,}39}{t^{1{,}8}}$	$1-\frac{1}{3{,}29-1{,}44t^{0{,}09}}$

Cuadro 4.2: Various fractional expressions for energy density, pressure and radiation EoS rounded to two decimal places. In $(\alpha,\beta)\to(1,1)$, $\beta\to1$ is taken before $\alpha\to1$

Explicit results for some α,β close to one are presented in Tab.4.2 to better appreciate the change in energy density, pressure and EoS.

4.2. Solutions in $3+1$ dimensional Cosmology

Building on the cosmological results in $2+1$ dimensions derived in the previous section, we observe that adopting Cartesian coordinates allows us to circumvent the r^2 angular scaling factor in the metric, yielding a more straightforward solution. Similarly, in $3+1$ dimensional cosmology, the coordinate transformation circumvents not only the angular scaling r^2 but also resolves the "polar angle problemïntroduced by the $\sin^2\theta$ term and the complexities that arise from applying the fractional Leibniz's rule to trigonometric functions, as discussed in the previous chapter.

This simplification arises due to the convenient structure of the cosmological metric in the case of a spatially flat geometry ($k = 0$), where the spatial part of the metric depends solely on the scaling factor $a(t)$, in contrast to standard polar coordinates (see appendix B). In the application of fractional calculus to FLRW equations with spatially flat geometry in 3+1 dimensions the calculations are thus greatly simplified.

$$ds^2 = dt^2 - A(t)\left(dx^2 + dy^2 + dz^2\right), \tag{4.29}$$

where $A(t) = a(t)^2$ is the squared scale factor. Applying the notation (2.20) convention maintains that the index α is used in the definition of the Christoffel symbols

$$\begin{aligned}\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} &= \frac{d^\alpha A}{2} \\ \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} &= \frac{d^\alpha A}{2A}.\end{aligned} \tag{4.30}$$

and β in the definition of the Riemann tensor

$$\begin{aligned}R^0_{110} = R^0_{220} = R^0_{330} &= -\frac{d^\beta(d^\alpha A)}{2} + \frac{(d^\alpha A)^2}{4A} \\ R^1_{010} = R^2_{020} = R^3_{030} &= -\frac{d^\beta\left(A^{-1}d^\alpha A\right)}{2} - \frac{(d^\alpha A)^2}{4A^2} \\ R^1_{221} = R^1_{331} = R^2_{332} &= -\frac{(d^\alpha A)^2}{4A} \\ R^2_{121} = R^3_{131} = R^3_{232} &= \frac{(d^\alpha A)^2}{4A},\end{aligned} \tag{4.31}$$

where $\alpha < \beta < 1$. With these considerations, the fractional Einstein field equations are succinctly expressed

$$T_0^0 = -\frac{3d^\beta\left(A^{-1}d^\alpha A\right)}{32\pi} + \frac{3d^\beta\left(d^\alpha A\right)}{32\pi A} \tag{4.32}$$

$$T_1^1 = T_2^2 = T_3^3 = \frac{(d^\alpha A)^2}{16\pi A^2} + \frac{3d^\beta\left(A^{-1}d^\alpha A\right)}{32\pi} + \frac{d^\beta\left(d^\alpha A\right)}{32\pi A}. \tag{4.33}$$

The next step is to solve the set of fractional derivatives (4.32) and (4.33). We assign $T_\nu^\mu = (\rho, -p, -p, -p)$ with ρ, p, as energy density and pressure.

4.2.1. Matter-dominated Universe

The scale factor for matter-dominated 3+1 Universe is proportional to $t^{2/3}$ [44]. Thus, the squared scale factor is $A \propto t^{4/3}$. Taking, for simplicity, $A = \left(\frac{t}{t_0}\right)^{4/3}$ with reference time t_0, and solving (4.32) and (4.33) results in

$$\rho(t) = \frac{t^{-\alpha-\beta}\alpha\Gamma(\frac{7}{3})\Gamma(1-\alpha)}{8\pi\Gamma(\frac{10}{3}-\alpha)\Gamma(2-\alpha-\beta)} + \frac{t^{-\alpha-\beta}(3\alpha-4)\Gamma(\frac{7}{3})}{8\pi(3\alpha-7)\Gamma(\frac{10}{3}-\alpha-\beta)} \tag{4.34}$$

$$p(t) = -\frac{t^{-2\alpha}\Gamma(\frac{7}{3})^2}{9\pi\Gamma(\frac{10}{3}-\alpha)^2} + \frac{t^{-\alpha-\beta}\alpha\Gamma(\frac{7}{3})\Gamma(1-\alpha)}{8\pi\Gamma(\frac{10}{3}-\alpha)\Gamma(2-\alpha-\beta)} - \frac{t^{-\alpha-\beta}(3\alpha-4)\Gamma(\frac{7}{3})}{24\pi(3\alpha-7)\Gamma(\frac{10}{3}-\alpha-\beta)}. \tag{4.35}$$

When $\alpha, \beta \to 1$ the classic energy density and pressure for $3+1$ matter-dominated Universe are recovered.

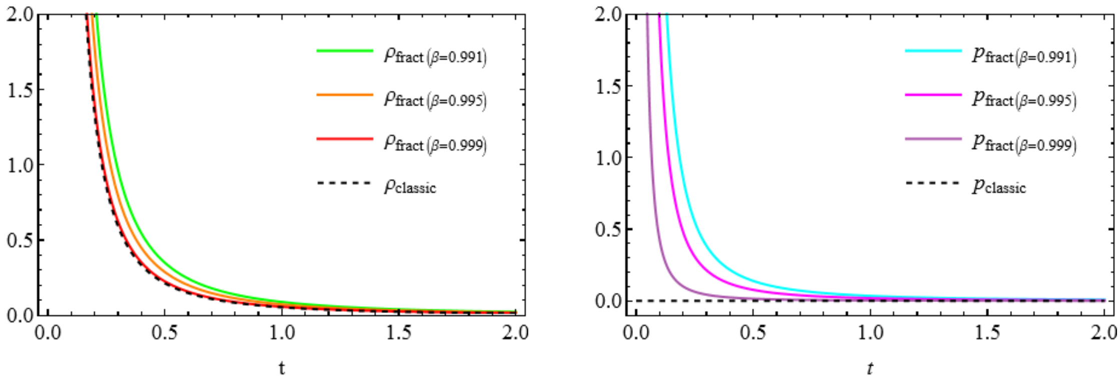
$$\lim_{\alpha,\beta\to 1} \rho(t) = \frac{1}{6\pi t^2} \tag{4.36}$$

$$\lim_{\alpha,\beta\to 1} p(t) = 0 \tag{4.37}$$

in natural units. The series expansion in (4.34) and (4.35) around the classic limit $\alpha = 1$ and $\beta = 1$ yield

$$
\begin{aligned}
\rho(t) &= \frac{1}{6\pi t^2} + \frac{\beta - 1}{8\pi t^2(\alpha - 1)} + O\left(\alpha - 1, \beta - 1\right) && (4.38)\\
p(t) &= \frac{\beta - 1}{8\pi t^2(\alpha - 1)} + O\left(\alpha - 1, \beta - 1\right), && (4.39)
\end{aligned}
$$

maintaining the hierarchy requirement of $\alpha < \beta$ as α, $\beta \to 1$ necessary for the classic convergence. As in the $2+1$ model the inclusion of non-locality induces a small pressure within the dust and results in a subtle alteration of the density profile. Full fractional expressions $\rho(t)$ and $p(t)$ are plotted in Fig.4.3 under this prescription.



(a) Density (b) Pressure

Figura 4.3: Fractional energy density (a) and pressure (b) functions with respect to time for a $3+1$ matter-dominated Universe. α is fixed at 0,99 and β varies.

Similar to the lower dimensional model, Fig.4.3a and Fig.4.3b, show a departure from classic density and pressure as $\beta \to \alpha$ and fractional parameters get closer. In contrast, $\beta \to 1$ reduces the deviation. It seems non-locality by the fractional operator similarly affects energy density and pressure values for small t in $3+1$

dimensions. Notably an apparent non-negligible dust pressure plays a role, specially in the early Universe.

The non-local nature of the theory impacts the ideal cosmic fluid and its EoS. The EoS $p = 0$ for dust would be henceforth altered. Specifically, the proportionality constant ω is now, according to (4.34) and (4.35) time dependant with dependency

$$\omega(t) = 1 - \frac{4t^{-\alpha-\beta}(3\alpha-4)\Gamma(\frac{7}{3})}{\rho(t)24\pi(3\alpha-7)\Gamma(\frac{10}{3}-\alpha-\beta)} - \frac{t^{-2\alpha}\Gamma(\frac{7}{3})^2}{\rho(t)9\pi\Gamma(\frac{10}{3}-\alpha)^2} \tag{4.40}$$

where in the classic limit $\alpha, \beta \to 1$ the EoS recovered.

$$\lim_{\alpha,\beta\to 1} \omega(t) \quad = \quad 0. \tag{4.41}$$

By considering the expansions (4.38) and (4.39) near the classical limit,

$$\omega = 1 - \frac{1}{1+\frac{6}{8}\left(\frac{\beta-1}{\alpha-1}\right)} + O\left(\alpha-1, \beta-1\right), \tag{4.42}$$

implying the relationship between density and pressure is highly sensitive to the fractional parameters. To achieve convergence towards the classical solution, it is again crucial that β approaches 1 before α.

However, it remains the case that outside this limit $\omega(t)$ is time-dependant, with an slightly evolving EoS over time. Similarly, a non-negligible dust interaction stands out and decays over time. Explicit $\rho(t)$ and $p(t)$ expressions for some α, β are presented in Tab.4.3 regarding the change in energy density, pressure and EoS.

Dust $(3+1)$	$a(t)$	$8\pi\rho(t)$	$8\pi p(t)$	$\omega(t)$
$(\alpha,\beta)\to(1,1)$	$t^{2/3}$	$\frac{4}{3t^2}$	0	0
$(\alpha,\beta)=(0{,}9,0{,}91)$	$t^{2/3}$	$\frac{2{,}06}{t^{1{,}81}}$	$\frac{1{,}52}{t^{1{,}81}}$-$\frac{0{,}78}{t^{1{,}8}}$	$0{,}74-0{,}38t^{0{,}01}$
$(\alpha,\beta)=(0{,}9,0{,}95)$	$t^{2/3}$	$\frac{1{,}7}{t^{1{,}85}}$	$\frac{1{,}15}{t^{1{,}85}}$-$\frac{0{,}78}{t^{1{,}8}}$	$0{,}68-0{,}46t^{0{,}05}$
$(\alpha,\beta)=(0{,}9,0{,}99)$	$t^{2/3}$	$\frac{1{,}34}{t^{1{,}89}}$	$\frac{0{,}8}{t^{1{,}89}}$-$\frac{0{,}78}{t^{1{,}8}}$	$0{,}59-0{,}58t^{0{,}09}$

Cuadro 4.3: Various fractional expressions for energy density, pressure and dust EoS rounded to two decimal places. In $(\alpha,\beta)\to(1,1)$, $\beta\to 1$ is taken before $\alpha\to 1$.

The presented solution highlights intriguing deviations from the standard cold dark matter (CDM) model, traditionally characterized by pressureless dust ($p = 0$). The apparent nonlocality, arising from the fractional operator and tuned with the parameters α, β, affects both energy density and pressure values proportionally. These effects might align with current cosmological observations, which increasingly supports deviations from the traditional assumption on the "coldness" of dark matter. For instance, recent studies derive an equation of state for dark matter [45, 46] where constrains from observational data impose a non-zero pressure component.

4.2.2. Radiation-dominated Universe

The scale factor for radiation dominated $3+1$ Universe is proportional to $t^{1/2}$ [44]. Thus, the squared scale factor is $A \propto t$. For instance, taking $A = \frac{t}{t_0}$ with reference time t_0, and solving (4.32) and (4.33) gives

$$\rho(t) = \frac{3t^{-\alpha-\beta}\alpha}{32\pi(\alpha^2-3\alpha+2)\Gamma(2-\alpha-\beta)} + \frac{3t^{-\alpha-\beta}(\alpha-1)}{32\pi(\alpha-2)\Gamma(3-\alpha-\beta)} \tag{4.43}$$

$$p_r(t) = -\frac{t^{-2\alpha}}{16\pi\Gamma(3-\alpha)^2} + \frac{3t^{-\alpha-\beta}\alpha}{32\pi(\alpha^2-3\alpha+2)\Gamma(2-\alpha-\beta)} - \frac{t^{-\alpha-\beta}(\alpha-1)}{32\pi(\alpha-2)\Gamma(3-\alpha-\beta)}. \tag{4.44}$$

When $\alpha, \beta \to 1$ the classic energy density and pressure for $3+1$ radiation-dominated Universe are recovered.

$$\lim_{\alpha,\beta\to 1} \rho(t) = \frac{3}{32\pi t^2} \tag{4.45}$$

$$\lim_{\alpha,\beta\to 1} p(t) = \frac{1}{32\pi t^2}, \tag{4.46}$$

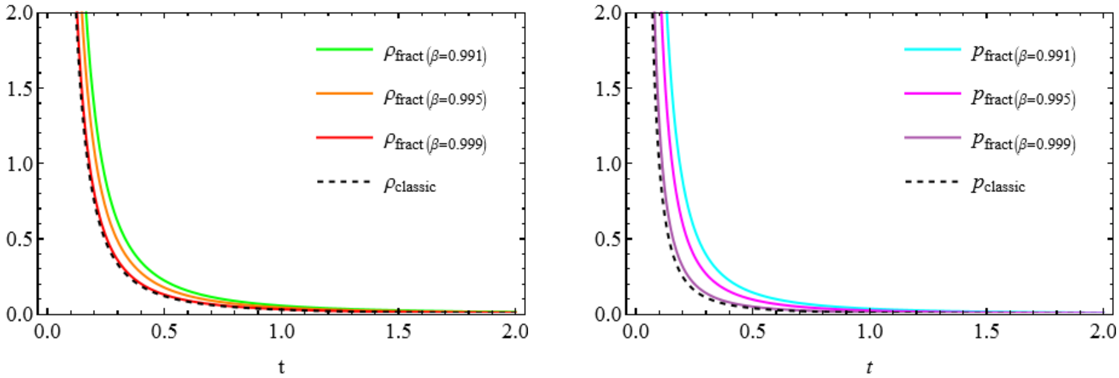
in natural units. The series expansion in (4.43) and (4.44) around the classic limit $\alpha = 1$ and $\beta = 1$ yield

$$\rho(t) = \frac{3}{32\pi t^2} + \frac{3(\beta-1)}{32\pi t^2(\alpha-1)} + O\left(\alpha-1, \beta-1\right) \tag{4.47}$$

$$p(t) = \frac{1}{32\pi t^2} + \frac{3(\beta-1)}{32\pi t^2(\alpha-1)} + O\left(\alpha-1, \beta-1\right), \tag{4.48}$$

maintaining the hierarchy requirement upon α and β to converge to (4.45) and (4.46). Full fractional expressions $\rho(t)$ and $p(t)$ are plotted in Fig.4.4 under this consideration.

Analogous to the matter-dominated case Fig.4.4a and Fig.4.4b depict how variations in the fractional parameters for the Christoffel symbols and curvature impact the energy density, consistent with expectations from the $2+1$ radiation-



(a) Density (b) Pressure

Figura 4.4: Fractional energy density (a) and pressure (b) functions with respect to time for a $3+1$ radiation-dominated Universe. α is fixed at 0,99 and β varies.

dominated model.

Again, considering the properties of the cosmic fluid altered by the non-local nature of the theory, the ideal EoS $p = \frac{1}{3}$ for radiation varies. Specifically, the proportionality constant ω is, according to (4.43) and (4.44)

$$\omega(t) = 1 - \frac{t^{-\alpha-\beta}(3\alpha-4)\Gamma(\frac{7}{3})}{\rho(t)6\pi(3\alpha-7)\Gamma(\frac{10}{3}-\alpha-\beta)} - -\frac{t^{-2\alpha}\Gamma(\frac{7}{3})^2}{\rho(t)9\pi\Gamma(\frac{10}{3}-\alpha)^2} \tag{4.49}$$

time dependant as well, with classic limit $\alpha, \beta \to 1$ recovering the EoS

$$\lim_{\alpha,\beta\to 1} \omega(t) \;=\; \frac{1}{3}. \tag{4.50}$$

Using the expansions (4.47) and (4.48) near the classical limit,

$$\omega = 1 - \frac{2}{3+3\left(\frac{\beta-1}{\alpha-1}\right)} + O\left(\alpha-1, \beta-1\right), \tag{4.51}$$

also shows that the density-pressure relationship is affected by the fractional parameters, and that β must approach 1 before α to ensure classical convergence.

Radiation $(3+1)$	$a(t)$	$8\pi\rho(t)$	$8\pi p(t)$	$\omega(t)$
$(\alpha,\beta)\to(1,1)$	$t^{1/2}$	$\frac{3}{4t^2}$	$\frac{1}{4t^2}$	$\frac{1}{3}$
$(\alpha,\beta)=(0{,}9,0{,}91)$	$t^{1/2}$	$\frac{1{,}34}{t^{1{,}81}}$	$\frac{1{,}24}{t^{1{,}81}}$-$\frac{0{,}46}{t^{1{,}8}}$	$0{,}93-0{,}34t^{0{,}01}$
$(\alpha,\beta)=(0{,}9,0{,}95)$	$t^{1/2}$	$\frac{1{,}06}{t^{1{,}85}}$	$\frac{0{,}96}{t^{1{,}85}}$-$\frac{0{,}46}{t^{1{,}8}}$	$0{,}90-0{,}43t^{0{,}05}$
$(\alpha,\beta)=(0{,}9,0{,}99)$	$t^{1/2}$	$\frac{0{,}78}{t^{1{,}89}}$	$\frac{0{,}69}{t^{1{,}89}}$-$\frac{0{,}46}{t^{1{,}8}}$	$0{,}88-0{,}58t^{0{,}09}$

Cuadro 4.4: Various fractional expressions for energy density, pressure and radiation EoS rounded to two decimal places. In $(\alpha,\beta)\to(1,1)$, $\beta\to 1$ is taken before $\alpha\to 1$.

Explicit $\rho(t)$ and $p(t)$ expressions for some α,β are presented in Tab.4.4 exemplifying the change in energy density, pressure and EoS.

The presented solution introduces intriguing modifications to the standard description of the matter sector during the radiation-dominated era, where the dynamics are typically characterized by well-established power laws governing the evolution of density perturbations. The incorporation of nonlocality, through a fractional operator parameterized by α and β slightly alters these traditional power laws by introducing time proportional changes to energy density and pressure. Although radiation's contribution to the late-time evolution of the universe is negligible, cosmological models incorporating power-law corrections provide a compelling fit to recent observational data [47]. These findings suggest the necessity of revisiting the standard framework, particularly during earlier epochs when radiation played a dominant role in shaping the dynamics of cosmic expansion.

Capítulo 5

Conclusions

The application of fractional calculus in General Relativity presents a challenge not only for theoretical physics but also for applied mathematics, making it an area worth investigating because of its potential to address theoretical issues. Additionally, with astronomical and cosmological observations reaching high precision levels, there is now an opportune time to test theories beyond General Relativity. With this potential in mind, our work was dedicated to exploring the feasibility of formulating a set of fractional Einstein field equations for a specific parameterization of the metric, similar to the approach used with classical derivative operators.

To accomplish this, we examined the general metric of a static and circularly symmetric 2+1-dimensional spacetime to derive the set of specific equations. However, we encountered a challenge when using either the Riemann-Lioville or Caputo operators, rendering it impossible to obtain the equations for solving. To address this issue, we introduced a weighted Riemann-Liouville derivative. The introduction of weights into the derivative serves a dual purpose. First, it enables the definition of a Riemann-Liouville derivative such that the derivative of a constant is zero, mirroring classical derivations. This is crucial in constructing Einstein equations, as utilizing the standard definition of the Riemann-Liouville operator could result in equations depending on undesired coordinates. For example, a static spacetime metric might yield equations depending on time owing to the standard Riemann-Liouville derivative of a constant not being zero. Building upon this new definition, we derived a set of integro-differential equations to solve.

As a specific example, we investigated the static Bañados-Teitelboim-Zanelli metric, a vacuum solution of the Einstein field equations with a negative cosmological constant, to determine if it also satisfies the fractional equations. We find that the solution for fractional parameters close to one leads to a solution with an anisotropic cosmological constant and an effective charge. In other words, the solution is similar to that of a charged BTZ, but with an anisotropic cosmological constant. Based on this result, we conclude that the non-locality introduced by the fractional parameters acts as a kind of Kaluza-Klein mechanism. Exploring the implications of our findings in a cosmological scenario could yield valuable insights, particularly regarding their alignment with the cosmological observations.

Cosmology in flat space-geometry yields another fruitful example since a convenient change of variables allows for $3 + 1$ fractional equations. We find that introducing fractional derivatives from the outset influences the evolution of cosmic species' matter sector. This framework modifies the standard cosmological model, with decelerated power-law behaviors, which can be fine-tuned by carefully selecting the fractional parameters. These findings merit further research, combining observational data to determine whether a couple of fractional parameters close to one can account for discrepancies in the standard model. Considering that the classic results are fully recovered as $\alpha, \beta \to 1$, allowing for an emergent theory of General Relativity.

Bibliografía

[1] G. Calcagni, L. Rachwał, Ultraviolet-complete quantum field theories with fractional operators, JCAP 09 (2023).

[2] G. Calcagni, Multifractional theories: an unconventional review, JHEP 03 (2017) 138.

[3] G. Calcagni, Multifractional theories: an updated review, Mod. Phys. Lett. A 36 (2021) 2140006.

[4] G. Calcagni, Classical and quantum gravity with fractional operators, Class. Quantum Gravity 38 (2021) 165005.

[5] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and applications of fractional differential equations. North-Holland Mathematics Studies vol. 204, Elsevier, Amsterdam (2006)

[6] Podlubny, I.: Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering. Academic Press, San Diego (1999)

[7] Samko, S.G., Kilbas, A.A., Marichev, O,I.: Fractional integrals and derivatives: theory and applications. Gordon and Breach, New York (1993).

[8] J.D. Barrow, A.B. Burd and D. Lancaster, Three-dimensional classical spacetimes, Class. Quantum Grav. 3 (1986) 551-567.

[9] N.J. Cornish and N.E. Frankel, Gravitation in 2+1 dimensions, Physical Review D. Vol. 43, No. 8 (1991).

[10] T. Dereli and R.W. Tucker, Gravitational interactions in 2+1 dimensions, Class. Quantum Grav. 5 (1988) 951-959.

[11] S Carlip, Conformal field theory, (2 + 1)-dimensional gravity and the BTZ black hole, Class. Quantum Grav. 22 R85 (2005).

[12] J. Maldacena, The Large N Limit of Superconformal Field Theories and Supergravity, Adv. Theor. Math. Phys. 2 (1998).

[13] S. Wu and S. Wei, Thermodynamical topology of quantum BTZ black hole, Phys. Rev. D 110, 024054 (2024).

[14] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).

[15] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D 48, 1506 (1993).

[16] C. Martínez, C. Teitelboim, J. Zanelli, Phys. Rev. D, **61**, 1040130 (2000).

[17] J. M. Overduin and P. S. Wesson, Phys. Rept. **283** (1997), 303-380 doi:10.1016/S0370-1573(96)00046-4 [arXiv:gr-qc/9805018 [gr-qc]].

[18] T. Mei, On isotropic metric of Schwarzschild solution of Einstein equation. arXiv preprint (2006): gr-qc/0610112.

[19] N. Lamprou et al, Class. Quantum Grav. 29 025002, (2012)

[20] Rubin, V.C., Ford, W.K. Jr., Thonnard, N. (1980), ApJ, 238, 471.

[21] Salucci, P. Dark Matter in Galaxies: Evidences and Challenges. Found Phys 48, 1517–1537 (2018).

[22] G. Gentile, P. Salucci, U. Klein, D. Vergani, P. Kalberla, The cored distribution of dark matter in spiral galaxies, Monthly Notices of the Royal Astronomical Society, Volume 351, Issue 3, July 2004, Pages 903–922.

[23] Markevitch, M., Gonzalez, A. H., Clowe, D., et al. (2004), ApJ, 606, 819

[24] Natarajan, A. (2013). Bounds on Dark Matter from CMB Observations. In: Cline, D. (eds) Sources and Detection of Dark Matter and Dark Energy in the Universe. Springer Proceedings in Physics, vol 148. Springer.

[25] Madhavacheril, M et al, Evidence of Lensing of the Cosmic Microwave Background by Dark Matter Halos, PRL 114, 151302 (2015).

[26] M. Mirzakhani and S. Maludze, Important Results of Different Experiments in Searching for Dark Matter Using Germanium and Silicon Detectors: A Comprehensive Review for Detecting Weakly Interacting Massive Particles, arXiv:2409.08900 (2024).

[27] Adhikari, R., Agostini, M., Ky, N. A., et al. (2017), JCAP, 1, 25.

[28] Salucci, P., Esposito, G., Lambiase, G., et al. (2021), Front. Phys.,8, 603190.

[29] E. Gonzales et al, Exact Solutions and Cosmological Constraints in Fractional Cosmology, Fractal Fract. (2023), 7, 368.

[30] G. Leon et al, Cosmology under the fractional calculus approach: a possible H_0 tension resolution?, PoS (2022) 248.

[31] M. García-Aspeitia, G. Fernandez-Anaya, A. Hernández-Almada, G. Leon, J. Magaña, Cosmology under the fractional calculus approach, Monthly Notices of the Royal Astronomical Society, Volume 517, Issue 4, (2022) 4813–4826.

[32] B. Milcota-Riascos et al, Revisiting Fractional Cosmology, Fractal Fract. (2023), 7, 149.

[33] G. Rajni, M. Ekta, M. Mudita, Bianchi Type-II Fractional Cosmological Model with Variable Lambda, (2022).

[34] A.R. El-nabulsi, Fractional Unstable Euclidean Universe, EJTP 8 (2005) 1–11.

[35] A.R. El-nabulsi, Cosmology with a Fractional Action Principle, Romanian Reports in Physics, Vol. 59, No. 3, P. 763–771, (2007).

[36] Barrientos, E.; Mendoza, S.; Padilla, P. Extending Friedmann Equations Using Fractional Derivatives Using a Last Step Modification Technique: The Case of a Matter Dominated Accelerated Expanding Universe. Symmetry (2021) 13, 174.

[37] F. Bennetti et al, Dark Matter in Fractional Gravity I: Astrophysical Tests on Galactic Scales, The Astrophysical Journal, Volume 949, Number 2 (2023).

[38] N. Cruz, C. Martinez, Cosmological scaling solutionsof minimally coupled scalar fields in three dimensions, Class. Quantum Grav. 17 (2000) 2867–2874.

[39] F.L. Williams and P.G. Kevrekidis, On (2 + 1)-dimensional Friedmann–Robertson–Walker universes: an Ermakov–Pinney equation approach, Class. Quantum Grav. 20 (2003) 177-184.

[40] G.S. Sharov, Multidimensional cosmological solutions of Friedmann type, Theoretical and Mathematical Physics. Vol. 101, No. 3, (1994).

[41] W.C. Saslaw, A relation between the homogeneity of the Universe and the dimensionality of space, Mon.Not. R. astr. Soc. (1977) 179, 659-662.

[42] B. Wang and E. Abdalla, Holography in 2ž / q1 -dimensional cosmological models, Physics Letters B. 466 (1999) 122–126.

[43] B. Wang, E. Abdalla and T. Osada, Entropy and holography constraints for inhomogeneous universes,Phys. Rev. Lett. 85, 5507 (2000).

[44] R.J. Alder, General Relativity and Cosmology, Springer, (2021).

[45] J. Barranco, A. Bernal, D. Núñez, Dark matter equation of state from rotational curves of galaxies, Monthly Notices of the Royal Astronomical Society, Volume 449, Issue 1, Pages 403–413, (2015).

[46] Y. Yao, J. Wang and X. Meng, Observational constraints on noncold dark matter and phenomenological emergent dark energy, Physical Review D, (2023).

[47] M. Koussour, N.S. Kavya, K. Venkatesha and N. Myrzakulov, Cosmic expansion beyond ΛCDM: Investigating power-law and logarithmic corrections. The European Physical Journal Plus, 139(2), (2024).

Capítulo 6

Apéndices

A. Solutions in $2+1$ Standard Metric

In what follows, we obtain the explicit form of the fractional Einstein field equations for the $2+1$ dimension cosmological standard flat metric

$$ds^2 = dt^2 - A(t)\left(dr^2 + R(t)d\theta^2\right) \tag{6.1}$$

where $A(t) = a(t)^2$ is the squared scale factor. The power rule in functions A, R is hidden so that classic Leibniz rule is not unintentionally applied in the context of fractional calculus. The convention is that the index α is used in the definition of the Christoffel symbols and β in the definition of the Riemann tensor. Note that, because the metric is a function of the radial as well as time coordinate, there will be radial and time partial fractional derivatives by the notation (2.20) d_r^α and d_t^α.

$$T_0^0 = -\frac{(d_t^\alpha A)^2}{8\pi A^2} - \frac{d_t^\beta\left(A^{-1}d_t^\alpha A\right)}{4\pi} + \frac{d_t^\beta\left(d_t^\alpha A\right)}{4\pi A} - \frac{F}{A} \tag{6.2}$$

$$T_1^1 = \frac{(d_t^\alpha A)^2}{8\pi A^2} + \frac{d_t^\beta\left(A^{-1}d_t^\alpha A\right)}{4\pi} + \frac{G}{A} \tag{6.3}$$

$$T_2^2 = \frac{(d_t^\alpha A)^2}{8\pi A^2} + \frac{d_t^\beta\left(A^{-1}d_t^\alpha A\right)}{4\pi} - \frac{G}{A} \tag{6.4}$$

where F, G are functions depending only on the radial coordinate r

$$F = \frac{d_r^\beta\left(R^{-1}d_r^\alpha R\right)}{8\pi} + \frac{d_r^\beta(d_r^\alpha R)}{8\pi R} \tag{6.5}$$

$$G = \frac{\left(d_r^\alpha R\right)^2}{8\pi R^2} + \frac{d_r^\beta\left(R^{-1}d_r^\alpha R\right)}{8\pi} - \frac{d_r^\beta(d_r^\alpha R)}{8\pi R}. \tag{6.6}$$

With the property that taking $R = r^2$ as usual and then the classic limit $\alpha, \beta \to 1$ results in both $F, G \to 0$, agreeing with the classic solution, which has no radial dependency. However, outside the limit, the solutions are radial dependant. See for instance, the matter-dominated Universe case $A \propto t^2$ in (6.2)-(6.4)

$$\begin{aligned} \rho(r,t) &= -\frac{2t^{-2\alpha}}{\pi\Gamma(4-\alpha)^2} + \frac{t^{-\alpha-\beta}\alpha\Gamma(1-\alpha)}{\pi\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)} + \frac{t^{-\alpha-\beta}(\alpha-2)}{\pi(\alpha-3)\Gamma(4-\alpha-\beta)} \\ &\quad + \frac{r^{-\alpha-\beta}\alpha\Gamma(1-\alpha)}{2\pi t^2\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)} - \frac{r^{-\alpha-\beta}(\alpha-2)}{2\pi t^2(\alpha-3)\Gamma(4-\alpha-\beta)} \end{aligned} \tag{6.7}$$

$$\begin{aligned} p_r(r,t) &= -\frac{2t^{-2\alpha}}{\pi\Gamma(4-\alpha)^2} + \frac{t^{-\alpha-\beta}\alpha\Gamma(1-\alpha)}{\pi\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)} - \frac{2r^{-2\alpha}}{\pi t^4\Gamma(4-\alpha)^2} \\ &\quad + \frac{r^{-\alpha-\beta}\alpha\Gamma(1-\alpha)}{2\pi t^2\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)} + \frac{r^{-\alpha-\beta}(\alpha-2)}{2\pi t^2(\alpha-3)\Gamma(4-\alpha-\beta)} \end{aligned} \tag{6.8}$$

$$\begin{aligned} p_t(r,t) &= -\frac{2t^{-2\alpha}}{\pi\Gamma(4-\alpha)^2} + \frac{t^{-\alpha-\beta}\alpha\Gamma(1-\alpha)}{\pi\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)} + \frac{2r^{-2\alpha}}{\pi t^4\Gamma(4-\alpha)^2} \\ &\quad - \frac{r^{-\alpha-\beta}\alpha\Gamma(1-\alpha)}{2\pi t^2\Gamma(4-\alpha)\Gamma(2-\alpha-\beta)} - \frac{r^{-\alpha-\beta}(\alpha-2)}{2\pi t^2(\alpha-3)\Gamma(4-\alpha-\beta)}. \end{aligned} \tag{6.9}$$

The application of the fractional derivative of order α, β results in the time and radial dependant expressions (6.7)-(6.9) for energy density, radial pressure and tangential pressure. In the limit $\alpha, \beta \to 1$ the classic energy density and pressure for $2+1$ matter-dominated Universe are recovered without radial dependencies.

$$\begin{aligned} \lim_{\alpha,\beta\to 1} \rho(r,t) &= \frac{1}{2\pi t^2} && (6.10)\\ \lim_{\alpha,\beta\to 1} p_r(r,t) &= 0 && (6.11)\\ \lim_{\alpha,\beta\to 1} p_t(r,t) &= 0, && (6.12) \end{aligned}$$

Thus, fractional modifications to the Einstein field equations in standard metric also recover classic Cosmology. However, having solved the equations also for the Cartesian metric in section 4.1 with radial independent solutions suggests that the dependency of radius in (6.7)-(6.9) is merely an effect of the change of coordinates.

B. Solutions in $3+1$ Standard Metric

In what follows, we obtain the explicit form of the fractional Einstein field equations for the $3+1$ dimension cosmological FLRW flat metric

$$ds^2 = dt^2 - A(t)\left(dr^2 + R(t)d\theta^2 + R(t)S(\theta)d\phi^2\right) \tag{6.13}$$

where $A(t) = a(t)^2$ is the squared scale factor, $R(r) = r^2$ and $S(\theta) = sin^2(\theta)$. The power rule in functions A, R and the trigonometric function $S(\theta)$ is hidden so that classic Leibniz rule is not unintentionally applied in the context of fractional calculus. The convention maintains that the index α is used in the definition of the Christoffel symbols and β in the definition of the Riemann tensor. Note that there will be radial, time and angular partial fractional derivatives by the notation (2.20) d_r^α, d_t^α and d_θ^α.

$$
\begin{aligned}
T_0^0 &= -\frac{3d_t^\beta\left(A^{-1}d_t^\alpha A\right)}{32\pi} + \frac{3d_t^\beta\left(d_t^\alpha A\right)}{32\pi A} - \frac{F}{A} - \frac{K}{AR} && (6.14)\\
T_1^1 &= \frac{(d_t^\alpha A)^2}{16\pi A^2} + \frac{3d_t^\beta\left(A^{-1}d_t^\alpha A\right)}{32\pi} + \frac{d_t^\beta\left(d_t^\alpha A\right)}{32\pi A} + \frac{G}{A} - \frac{L}{AR} && (6.15)\\
T_2^2 &= \frac{(d_t^\alpha A)^2}{16\pi A^2} + \frac{3d_t^\beta\left(A^{-1}d_t^\alpha A\right)}{32\pi} + \frac{d_t^\beta\left(d_t^\alpha A\right)}{32\pi A} - \frac{H}{A} + \frac{L}{AR} && (6.16)\\
T_3^3 &= \frac{(d_t^\alpha A)^2}{16\pi A^2} + \frac{3d_t^\beta\left(A^{-1}d_t^\alpha A\right)}{32\pi} + \frac{d_t^\beta\left(d_t^\alpha A\right)}{32\pi A} - \frac{H}{A} - \frac{L}{AR} && (6.17)
\end{aligned}
$$

where F, G and H are functions depending only on the radial coordinate r

$$
\begin{aligned}
F &= \frac{\left(d_r^\alpha R\right)^2}{32\pi R^2} + \frac{d_r^\beta\left(R^{-1}d_r^\alpha R\right)}{16\pi} + \frac{d_r^\beta(d_r^\alpha R)}{16\pi R} && (6.18)\\
G &= \frac{\left(d_r^\alpha R\right)^2}{32\pi R^2} + \frac{d_r^\beta\left(R^{-1}d_r^\alpha R\right)}{16\pi} - \frac{d_r^\beta(d_r^\alpha R)}{16\pi R} && (6.19)\\
H &= \frac{\left(d_r^\alpha R\right)^2}{32\pi R^2} + \frac{d_r^\beta\left(R^{-1}d_r^\alpha R\right)}{16\pi} && (6.20)
\end{aligned}
$$

and K and L are functions depending only on the polar angular coordinate θ

$$
\begin{aligned}
K &= \frac{d_\theta^\beta\left(S^{-1}d_t^\alpha S\right)}{32\pi} + \frac{d_\theta^\beta\left(d_\theta^\alpha S\right)}{32\pi S} && (6.21)\\
L &= \frac{(d_\theta^\alpha S)^2}{32\pi S^2} + \frac{d_\theta^\beta\left(S^{-1}d_t^\alpha S\right)}{32\pi} - \frac{d_\theta^\beta\left(d_\theta^\alpha S\right)}{32\pi S}. && (6.22)
\end{aligned}
$$

This set of equations remains unsolved for any scale factor. Simply taking $S = sin(\theta)^2$ is not trivial since the calculation involves infinite series whose convergence have not yet been demonstrated. Even if we conjecture all r and θ dependencies disappear in the classic limit as in $2+1$ dimensional Cosmology, it remains true that outside this limit fractional solutions will exhibit such dependencies inherited by the coordinate system.