UNIVERSIDAD SAN FRANCISCO DE QUITO USFQ

Colegio de Ciencias e Ingeniería

Electromagnetism in the Presence of Electric and Magnetic “Charges” in a Geometric Formulation: Dynamic Dyons

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Física

Trabajo de titulación presentado como requisito

para la obtención del titulo de

Físico

11 de mayo de 2025

UNIVERSIDAD SAN FRANCISCO DE QUITO USFQ

Colegio de Ciencias e Ingeniería

HOJA DE CALIFICACIÓN DE TRABAJO DE FIN DE CARRERA

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May 11, 2025

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ACLARACIÓN PARA LA PUBLICACIÓN

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Acknowledgements

I would like to express my deepest gratitude to all my professors at USFQ. They nurtured my curiosity and inspired me to keep exploring the mysteries of the universe. I am especially grateful to Ernesto Contreras, with whom I had the great pleasure of working and learning. He became more than a teacher—he became a friend.

I also want to thank Melissa Infusino and Edgar Carrera for their continuous support and for always fostering a healthy and welcoming environment for physics students. A special thanks goes to Carlos Marín—although his courses were challenging, they pushed me to grow and gave me confidence in my knowledge. To Aleksandr Novikov that showed me that equally important as being a good physicist is being a good person.

I am incredibly thankful to my father, who was always there for me, offering his unwavering support. Studying physics, while dealing with life's challenges, has not been easy, but the support of family and friends helped me through.

My heartfelt thanks go to A.P., my calm in the storm. You were the one who helped me the most during difficult times and always listened to my wild physics ideas. I will always be thankful for everything you taught me in these times. Finally, thanks to Daniel Paredes, a dear friend who was an essential part of this journey.

I will always be profoundly grateful to my teachers, my family and my friends for helping me achieve this extraordinary milestone and for making a dream come true. I owe it all to you.

Resumen

La motivación de este trabajo es explorar una representación geométrica para la cuantización de teorías de campos que produzcan naturalmente monopolos magnéticos. Dirac [1, 2] demostró que la existencia de monopolos magnéticos podría explicar la cuantización de la carga eléctrica. Aunque los monopolos magnéticos aún no han sido observados experimentalmente, su atractivo teórico sigue motivando el estudio de modelos que permiten su existencia. En particular, investigaciones sobre dyones—partículas que poseen carga eléctrica y magnética—dentro del marco de la Electrodinámica No Lineal (NLED, por sus siglas en inglés) han contribuido al desarrollo reciente de soluciones de Black Bounce (BB) a las ecuaciones de campo de Einstein, con el objetivo de resolver las singularidades centrales en los agujeros negros [3]. Por lo tanto, el estudio de teorías de campo que admiten monopolos magnéticos, y especialmente dyones, puede ofrecer valiosas ideas para abordar problemas abiertos en la física teórica contemporánea.

Palabras clave: *Monopolo magnético, Dyon, Representación geometrica, Teoría de campos, Cuantización canónica*

Abstract

The motivation behind this work is to explore a geometric representation for quantizing field theories that naturally give rise to magnetic monopoles. Dirac [1, 2] demonstrated that the existence of magnetic monopoles could explain the quantization of electric charge. Although magnetic monopoles have yet to be observed experimentally, their theoretical appeal continues to drive interest in models that permit their existence. In particular, research involving dyons—particles carrying both electric and magnetic charges—within the framework of Nonlinear Electrodynamics (NLED) has contributed to recent developments in Black Bounce (BB) solutions to Einstein's field equations, aiming to resolve central singularities in black holes [3]. Therefore, studying field theories that accommodate magnetic monopoles, and especially dyons, may offer valuable insights into longstanding open questions in theoretical physics.

Keywords: *Magnetic monopole, Dyon, Geometric representation, Field theory, Canonical quantization*

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Chapter 1

Introduction

The study of magnetic monopoles has long intrigued theoretical physicists, offering elegant explanations for fundamental phenomena such as the quantization of electric charge, as first proposed by Dirac [1, 2]. Although monopoles have not been experimentally observed, the theoretical interest remains strong—especially in more general configurations like *dyons*, particles that carry both electric and magnetic charge.

Dyons naturally arise in certain extensions of classical electrodynamics and in the study of dualities in field theory. Their inclusion enriches the gauge structure of the theory and provides deeper insight into topological and quantization phenomena. However, while much of the existing literature focuses on static configurations or single-charge systems, a consistent treatment of *dynamic dyons* - especially within a covariant and geometric framework - has remained largely unexplored.

A recent and intriguing motivation comes from gravitational physics. In 2018, Simpson and Visser proposed the *Black Bounce* (BB) model [4], a class of space-time solutions in General Relativity designed to resolve central singularities by introducing a smooth bounce in the radial coordinate. These models are often supported by matter content described by Nonlinear Electrodynamics (NLED), a framework originally introduced by Born and Infeld [5] which also accommodates magnetic monopoles and dyons. While initial BB solutions were built using purely magnetic sources, more recent work has incorporated *electric and dyonic* charges [3], broadening the scope of physically meaningful configurations.

These developments underscore the need for a rigorous and general formalism that can describe dyons, particularly in dynamic settings. This thesis addresses that need by extending the geometric representation of Maxwell theory to include dyonic particles in motion. Building on previous work in the geometric formulation of gauge theories [6, 7], we propose a framework that captures the dynamics of dyons while preserving the topological and covariant features of the theory.

The structure of this work is as follows. In the next section, we review the Lagrangian and Hamiltonian formulations of classical mechanics for systems with finite degrees of freedom. We then extend these ideas to systems with infinite degrees of freedom, setting the stage for field quantization. The canonical quantization procedure is introduced, along with the classification of first- and second-class constraints. Finally, we apply this formalism to the quantization of the first-order Schwinger action for dyonic particles, preparing the ground for the geometric formulation that follows in later chapters.

1.1 Hamiltonian formulation

Recall that the action is defined as:

$$S = \int_{t_1}^{t_2} dt L(q_i, \dot{q}_i), \tag{1.1}$$

Where q_i is the generalized coordinates and $\dot{q}_i$ the generalized velocities. Using the Hamilton principle also known as the principle of least action, varying the action to first order

$$\delta S = 0$$

we can deduce the Euler-Lagrange equations.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{1.2}$$

This is the well know Lagrangian formulation for analytical mechanics. If we define the conjugate momentum as follows:

$$p^i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{1.3}$$

We can perform a Legendre transformation to the Lagrangian to obtain the Hamiltonian, as follows:

$$H(q_i, p^i) = p^i \dot{q}_i - L(q_i, \dot{q}_i(p^i)) \tag{1.4}$$

Notice now that the Hamiltonian only depends of q_i and p^i. If we substitute the expression for the Lagrangian on the action (1.1) and use the principle of least action, we obtain the Hamilton equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p^i}, \qquad \dot{p}^i = -\frac{\partial H}{\partial q_i}, \tag{1.5}$$

We define for any functions A and B that live in phase space, the Poisson brackets as follows:

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial B}{\partial q_i} \tag{1.6}$$

Here we use the Einstein sum convention. It is easy to show that we can write Hamilton equations using the Poisson brackets, using the chain rule, we obtain:

$$\begin{aligned} \dot{q}_i &= \{q_i, H\} \\ \dot{p}^i &= \{p^i, H\} \end{aligned} \tag{1.7}$$

And in general for any quantity g the evolution of said quantity is given by

$$\dot{g} = \{g, H\}$$

Now notice that the equations of motion above are only valid if we can express the generalized velocities $\dot{q}^i$ in terms of (q_i, p^i), the variables of phase space.

1.1.1 Hamiltonian formulation for finite degrees of freedom

In the more general case, we will encounter relations of the type

$$\phi_m(q_i, p^j) = 0, \quad m = 1, 2, ..., M \tag{1.8}$$

where M is the number of such relations. This relations must be conserved over time and will be called primary constraints. The problem of primary constrains resolves using the method of Lagrange multipliers to the Hamiltonian, we define a new Hamiltonian H^*:

$$H^* = H \; + \; \lambda^m \phi_m \tag{1.9}$$

with λ^m being the Lagrange multipliers. If we follow the same procedure of using the principle of least action on the Hamiltonian H^*, we obtain after calculating the Poisson brackets and setting $\phi = 0$:

$$\begin{aligned} \dot{q}_i &= \{q_i, H^*\}, \\ \dot{p}^i &= \{p^i, H^*\}, \\ \phi_m(q_i, p^j) &= 0, \end{aligned} \tag{1.10}$$

It is important to set the constraints equal to zero only after all the Poisson brackets have been calculated. To take this into account Dirac introduced the notation $\phi \approx 0$, which reads "weakly equal". For example note that $H^* \approx H$.

Note that the evolution of any function g will now be given by:

$$\dot{g} = \{g, H^*\} = \{g, H\} + \lambda^m \{g, \phi_m\} \tag{1.11}$$

Now, primary constraints must be preserved in time that is:

$$\{\phi_m, H^*\} \approx \{\phi_m, H\} = 0 \tag{1.12}$$

While preserving in time primary constraints,

$$\dot{\phi}_m = \{\phi_m, H^*\} = \{\phi_m, H\} + \lambda^{m'} \{g, \phi_{m'}\} \tag{1.13}$$

four different cases might occur:

1. We obtain an identity $0 = 0$
2. We obtain an expression for the multipliers λ_m
3. We obtain new type of constraints $\tilde{\phi}_{\tilde{m}} \approx 0$ which are independent of primary constraints. This relations are called secondary constraints. Say there are $\tilde{M}$ of them.
4. We obtain an inconsistency

The process of preservation of constraints will not go on indefinitely it will end in option 1, 3, or 4. If new constraints appear we repeat the process until we get 1 or 3. If we obtain option 4, the theory must be discarded. At the end we will have $M + \tilde{M}$ constraints which for convenience we shall say we have M constrains in total.

Now we can define the extended Hamiltonian, which contains both primary and secondary constraints:

$$\tilde{H} = H \ + \ \lambda^m \phi_m \tag{1.14}$$

Using an extended variational principle the equations of motion can be written as follows:

$$\begin{aligned} \dot{q}_i &= \{q_i, \tilde{H}\}, \\ \dot{p}^i &= \{p^i, \tilde{H}\}, \end{aligned} \tag{1.15}$$

where automatically all constrains that preserve through time are set weakly to zero, $\phi_m \approx 0$. As done previously, time evolution for a function in phase space g is given by:

$$\dot{g} = \{g, \tilde{H}\} = \{g, H\} + \lambda^m \{g, \phi_m\} \tag{1.16}$$

Note that $\tilde{H} \approx H^* \approx H$. If we use the extended Hamiltonian to preserve the constraints we obtain:

$$\{\phi_m, H\} + \lambda^{m'} \triangle_{mm'} \approx 0 \tag{1.17}$$

where $\triangle_{mm'} = \{\phi_m, \phi_{m'}\}$, and note that they are the components of an $M \times M$ matrix. Also, $\{\phi_m, H\}$ and $\lambda^{m'}$ are column vectors of length M. The multipliers can be expressed as a sum of two terms:

$$\lambda^m = U^m + V^m$$

where $U^m = U^m(q_i, p^i)$ is a particular solution to the equation

$$U^m \{\phi_{m'}, \phi_m\} \approx -\{\phi_{m'}, H\}.$$

Meanwhile, $V^m = V^m(q_i, p^i)$ is a vector belonging to the kernel of the matrix $\triangle$ with components $\triangle_{mm'}$. That is,

$$V^{m'}\{\phi_m, \phi_{m'}\} \approx 0, \quad V^m \in \ker(\triangle).$$

If we define a basis $V_a^m = V_a^m(q_i, p^i)$ with $a = 1, \ldots, A$ in $\ker(\triangle)$, then we can write

$$V^m = v^a V_a^m.$$

Here, v^a are arbitrary coefficients. With this decomposition, the Hamiltonian $\tilde{H}$ takes the form

$$\tilde{H} = H + U^m\phi_m + v^a V_a^m \phi_m = H' + v^a\phi_a, \quad \text{where } \phi_a = V_a^m\phi_m. \tag{1.18}$$

The number of constraints ϕ_a matches the dimension of the kernel, meaning $\dim(\ker(\triangle)) = A$. Consequently, the time evolution of any phase space function g is given by

$$\dot{g} = \{g, H' + v^a\phi_a\}.$$

With this new notation, it follows that the Poisson bracket of H' with any constraint vanishes weakly. Specifically,

$$\{H', \phi_m\} = \{H, \phi_m\} + U^{m'}\{\phi_{m'}, \phi_m\} \approx 0. \tag{1.19}$$

Because of the preservation of all constraints and the fact that

$$\{\phi_a, \phi_m\} = V_a^{m'}\{\phi_{m'}, \phi_m\} \approx 0 \tag{1.20}$$

as V_a^m is an element of the kernel of $\triangle$. Note that there are $A = \dim(\ker(\triangle))$ number of first class constraints. The quantities whose Poisson bracket with all constraints is weakly zero are called **first class quantity**. Similarly if a quantity has at least one Poisson bracket that is not weakly zero with a constraint , then it is a **second class quantity**. Note that we also have made a rigorous definition for primary and secondary constraints. A constraint which Poisson bracket with all other constraints is weakly zero is a **first class constraint**, and following the same pattern we also define on the contrary **second class constraints**.

In general we have A primary constraints and $R = M - A$ secondary constraints which we will call χ_r. We can arrange the constraints in the following matrix:

$$\tilde{\triangle} \approx \begin{pmatrix} (\Lambda_{rr'})_{R\times R} & \{\chi_r, \phi'_a\}_{R\times A} \\ \{\phi_a, \phi_{r'}\}_{A\times R} & \{\phi_a, \phi'_a\}_{A\times A} \end{pmatrix} \approx \begin{pmatrix} (\Lambda_{rr'})_{R\times R} & 0_{R\times A} \\ 0_{A\times R} & 0_{A\times A} \end{pmatrix} \tag{1.21}$$

with $\Lambda_{rr'} \approx \{\chi_r, \chi_{r'}\}$. Dirac proved a theorem asserting that the determinant of the matrix with elements $\Lambda_{rr'} \approx \{\chi_r, \chi_{r'}\}$ does not vanish, even in a weak sense. Consequently, the number of second-class constraints (therefore, the matrix dimension) must be even.

We now define the Dirac brackets between two phase space functions F and G as

$$\{F, G\}^* = \{F, G\} - \{F, \phi_j\}\Delta^{jj'}\{\phi_{j'}, G\} \tag{1.22}$$

then the evolution equation can be written as

$$\dot{g} = \{g, H + v^a \phi_a\}^* \tag{1.23}$$

where we recall that $\{\phi_a, \chi_r\} \approx 0$. The Dirac brackets satisfy the same properties as Poisson brackets, namely

$$\{F, G\}^* = -\{G, F\}^* \tag{1.24}$$

$$\{F, GH\}^* = \{F, G\}^* H + G\{F, H\}^* \tag{1.25}$$

$$\{\{F, G\}^*, H\}^* + \{\{H, F\}^*, G\}^* + \{\{G, H\}^*, F\}^* = 0 \tag{1.26}$$

Additionally, the Dirac bracket of any phase space function with a second-class constraint χ_r is strongly zero. That is,

$$\begin{aligned}\{\chi_r, g\}^* &= \{\chi_r, g\} - \{\chi_r, \chi_{r'}\}\Delta^{r's'}\{\chi_{s'}, g\} \\ &= \{\phi_r, g\} - \Delta_{rr'}\Delta^{r's'}\{\chi_{s'}, g\} \\ &= \{\phi_r, g\} = 0\end{aligned}$$

Now, we turn our attention to a system of infinite degrees of freedom. That is, we want the Hamiltonian formulation for a field.

1.1.2 Hamiltonian formulation for infinite degrees of freedom

In this section, the degrees of freedom are now infinite. We assume that the reader has a basic understanding of the Lagrangian density

$$L = \int d^3x \mathcal{L}(\Phi_I, \partial_\mu \Phi_I)$$

where Φ_I is the field I, with $I = 1, 2, ..., N$, that is, N fields in general. The Lagrangian density must be considered as a function that contains all information of interest about the field we wish to study. Note now that L is a functional and we can write the action as

$$S = \int d^4x \mathcal{L}.$$

Now in the same way as for finite degrees of freedom, we can define the conjugate momentum for the field Φ_I as follows:

$$\Pi^I = \frac{\delta L}{\delta \dot{\Phi}_I} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_I} \tag{1.27}$$

where $\frac{\delta L}{\delta \dot{\Phi}_I}$ is the functional derivative with respect to the field $\dot{\Phi}_I$. We assume that the reader is familiar with this concept. Note that we supposed that Π^I can be written in terms of $\dot{\Phi}_I$, if not we will follow the same procedure as last section.

Now again by performing a Legendre transformation on $\mathcal{L}$, we define the **Hamiltonian density**:

$$\mathcal{H} = \left(\Pi^I \dot{\Phi}_I - \mathcal{L}\right)\Big|_{\Pi^I} \tag{1.28}$$

Again using the principle of least action one can show that the equations of motion can be written as:

$$\dot{\Phi}_I = \frac{\delta H}{\delta \Pi^I} = \frac{\partial \mathcal{H}}{\partial \Pi^I}, \tag{1.29}$$

$$\dot{\Pi}^I = -\frac{\delta H}{\delta \Phi_I} = -\frac{\partial \mathcal{H}}{\partial \Phi_I} + \partial_i \frac{\partial \mathcal{H}}{\partial(\partial_i \Phi_I)}, \tag{1.30}$$

Now we need to pass from the discrete formulation to the continuous, the Poisson Brackets are defined as

$$\{F, G\} = \int d^3x \left(\frac{\delta F}{\delta \Phi_I} \frac{\delta G}{\delta \Pi^I} - \frac{\delta F}{\delta \Pi^I} \frac{\delta G}{\delta \Phi_I} \right) \tag{1.31}$$

and thus the evolution in time will be given by

$$\dot{F} = \{F, H\}.$$

Now we want to calculate the Poisson brackets between the fields and their momenta, for that we define

$$\delta\Phi_I(\vec{x}, t) = \int d^3x' \frac{\delta\Phi_I(\vec{x}, t)}{\delta\Phi_J(\vec{x}', t)} \delta\Phi_J(\vec{x}', t) \tag{1.32}$$

from which necessarily

$$\frac{\delta\Phi_I(\vec{x}, t)}{\delta\Phi_J(\vec{x}', t)} = \delta_I^J \delta^3(\vec{x} - \vec{x}') \tag{1.33}$$

and in the same way we define

$$\frac{\delta \Pi^I(\vec{x},t)}{\delta \Pi^J(\vec{x}',t)} = \delta^I_J \delta^3(\vec{x}-\vec{x}'). \tag{1.34}$$

From this it is straightforward to calculate the Poisson brackets of the fields and momenta at **equal time**,

$$\{\Phi_I(\vec{x},t), \Pi^J(\vec{y},t)\} = \delta_I^J \delta^3(\vec{x}-\vec{y}) \tag{1.35}$$

$$\{\Phi_I(\vec{x},t), \Phi_J(\vec{y},t)\} = 0 \tag{1.36}$$

$$\{\Pi^I(\vec{x},t), \Pi^J(\vec{y},t)\} = 0 \tag{1.37}$$

Finally it is important to note that if we are dealing with a continuous system that has second class constraints, in the same way we must make the jump between discrete to continuous (i.e. from coordinates to fields, from sum to integration) and define the matrix $\Lambda_{rr'} \approx \{\chi_r, \chi_{r'}\}$, and then the Dirac brackets. We will not go into detail as the field we wish to quantize later on will only have primary constraints. Our next goal is to see how to quantize our field theory. In the same way, we will use the discrete case and then make the jump to the continuous. To quantize our theory we shall use the canonical quantization method.

1.2 Canonical quantization

Until now we have seen how to use the Hamiltonian formulation for out theory, now we want to follow some rules to quantize it. In the most simple case when there are no constraints we follow the next rules (setting $c = \hbar = 1$):

1. The canonical variables q_i, p^i now become operators $\hat{q}_i$, $\hat{p}^i$ that act on vectors $|\psi\rangle \in \mathbb{V}$, with $\mathbb{V}$ a Hilbert space.
2. The Poisson brackets are replaced by the commutators, that is

$$\{q_i, p^j\} \rightarrow -i[\hat{q}_i, \hat{p}^j]$$

   and with this we obtain the algebra of the operators.
3. The evolution of physical states is given by the Schrödinger equation

$$\hat{H}|\psi\rangle = i\frac{\partial}{\partial t}|\psi\rangle$$

This is the recipe to follow even when we have constraints, but we need to see what are the consequences of having such constraints in how we quantize our system.

1.2.1 Quantization of systems with only first class constraints

In the classical field theories, the first class constraints are the ones that generate the gauge transformations on the theory [8]. In the same way, we want to make first class constraints do the same thing to our quantize theory, making that for some representation of $\hat{\phi}_a$, obtain from converting the canonical variables into operators. We impose that $|\psi\rangle_{\text{phy}}$ are invariant under the transformation

$$e^{i\epsilon^a \hat{\phi}_a}|\psi\rangle_{\text{phy}} = 0 \tag{1.38}$$

which implies that

$$\hat{\phi}_a|\psi\rangle_{\text{phy}} = 0 \tag{1.39}$$

imposing constraints on the phase space. As a consequence, the procedure for quantizing theories with first-class constraints consists of the following:

1. The canonical variables q_i, p^i become operators $\hat{q}_i, \hat{p}^i$, which act on functionals $\psi[q,p] \in \mathbb{V}$, where $\mathbb{V}$ is a Hilbert space.

2. The algebra of operators is obtained from the Poisson brackets of the canonical variables:

$$\{q_i, p^j\} \to -i[\hat{q}_i, \hat{p}^j].$$

3. Physical functionals are those annihilated by the first-class constraints:

$$\hat{\phi}_a|\psi\rangle_{\text{phy}} = 0.$$

4. The evolution of the physical states follows from the Schrödinger equation:

$$\hat{\tilde{H}}|\psi\rangle_{\text{phy}} = \hat{H}'|\psi\rangle_{\text{phy}} + v^a\hat{\phi}_a|\psi\rangle_{\text{phy}} = \hat{H}|\psi\rangle_{\text{phy}} = i\frac{\partial}{\partial t}|\psi\rangle_{\text{phy}}.$$

Here, $\hat{H}$ is the Hamiltonian operator derived from the original Hamiltonian H, since H' includes (in this case) first-class constraints that annihilate the physical states.

1.3 Electromagnetism in the Presence of Electric and Magnetic "Charges"

As an example of all studied so far we will see how to follow the recipe outlined before using the Schwinger action. Our aim is to study a theory that allows the existence of particles with electric or magnetic "charges" and thus set the ground for studying particles who have both of them at the same time, that is a **dyons**. As Dirac postulated [1][2] the existence of a magnetic monopole will explain the quantization of electric charge in the universe, a mystery which is still unsolved to this day. Therefore there is still motivation to study such theories using the geometric representation. Our aim is to detail the previous work done regarding magnetic monopoles [6][9][7].

1.3.1 Schwinger action

We base our approach on the first-order Schwinger action[10], to which we add the presence of particles carrying electric charge and others carrying magnetic charge:

$$\begin{aligned} S = \int d^4x \Big(- A_\mu J_e^\mu - B_\mu J_m^\mu - \frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \Big) \\ + \int dt \left(\frac{1}{2} m \dot{\vec{r}}^2 + \frac{1}{2} M \dot{\vec{R}}^2 \right) \end{aligned} \tag{1.40}$$

where our canonical variables are given by the independent fields A_μ and $F^{\mu\nu}$. We also introduce variables associated with the position of the particles: $\vec{r}$ for those carrying electric charge ("small" particles) and $\vec{R}$ for those carrying magnetic charge (”large” particles). Moreover, J_e^μ and J_m^μ represent the currents corresponding to the electric and magnetic sources, respectively, given by:

$$J_e^\mu = eq_e v^\mu(t)\delta^{(3)}(\vec{x} - \vec{r}), \tag{1.41}$$

$$J_m^\mu = gQ_m v^\mu(t)\delta^{(3)}(\vec{x} - \vec{R}), \tag{1.42}$$

where e (g) represents a unit of electric (magnetic) charge flux, and q_e (Q_m) are integers representing the charge values carried by the particles at positions $\vec{r}$ (or $\vec{R}$). The velocity $v^\mu(t) = (1, \vec{v})$ allows us to compactly write the currents as $J_\xi^\mu = (\rho_\xi, \vec{J}_\xi)$. In Eq. (1.40), the field $B_\mu(x)$ is defined as:

$$B_\mu(x) = \int d^4y\, {}^*F^{\mu\nu}(y) f^\nu(y - x) + \partial_\mu \lambda_m(x), \tag{1.43}$$

where $\lambda_m(x)$ is an arbitrary function and $f^\nu(y)$ satisfies:

$$\partial_\nu f^\nu(y) = \delta^4(y) \tag{1.44}$$

In order to derive the equations of motion, we perform variations with respect to the independent fields A_μ and $F^{\mu\nu}$. The total variation of the action given in (4.1) then results in:

$$\delta S = \int d^4x \left(-\delta A_\mu J_e^\mu - \delta B_\mu J_m^\mu - \frac{1}{2}\delta F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \partial_\mu F^{\mu\nu}\delta A_\nu + \frac{1}{2}F^{\mu\nu}\delta F_{\mu\nu} \right)$$

By varying with respect to A_μ, we straightforwardly obtain:

$$\frac{\delta S}{\delta A_\mu} = -J_e^\mu + \partial_\nu F^{\nu\mu} = 0 \quad \Rightarrow \quad \partial_\nu F^{\nu\mu} = J_e^\mu. \tag{1.45}$$

Meanwhile, variations with respect to $F^{\mu\nu}$ contribute as follows:

$$\int d^4x \delta B_\alpha J_m^\alpha = \int d^4x \left(\int d^4y \delta\, {}^*F_{\alpha\beta}(y) f^\beta(y-x) \right) J_m^\alpha \tag{1.46}$$

$$= \int d^4x \left(\int d^4y \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} \delta F^{\mu\nu}(y) f^\beta(y-x) \right) J_m^\alpha, \tag{1.47}$$

which directly leads to:

$$\frac{\delta S}{\delta F^{\mu\nu}} = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} F_{\mu\nu} + \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} \int d^4y J_m^\alpha f^\beta(y-x) = 0. \tag{1.48}$$

Thus, we obtain:

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - \epsilon_{\mu\nu\lambda\sigma} \int d^4y J_m^\lambda(y) f^\sigma(y-x). \tag{1.49}$$

Taking the divergence of (1.49), contracting properly with the Levi-Civita antisymmetric tensor, and assuming the conservation of the current, we arrive at the second group of Maxwell's equations:

$$\partial_\nu {}^*F^{\mu\nu} = J_m^\mu. \tag{1.50}$$

In summary, incorporating the magnetic monopole modifies Maxwell's equations into the following general form:

$$\partial_\nu F^{\mu\nu} = J_e^\mu, \quad \partial_\nu {}^*F^{\mu\nu} = J_m^\mu, \tag{1.51}$$

where the dual field tensor is given by:

$$^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}. \tag{1.52}$$

Thus we now have magnetic and electric currents.

1.3.2 Canonical quantization of Schwinger action

We begin decomposing the action in $3+1$ dimensions:

$$\begin{aligned} S^{3+1} = \int d^4x \Big(& - A_0 J_e^0 - B_0 J_m^0 - A_i J_e^i - B_i J_m^i - F^{0i}(\partial_0 A_i - \partial_i A_0) \\ & - \frac{1}{2}F^{ij}(\partial_i A_j - \partial_j A_i) + \frac{1}{2}F^{0i}F_{0i} + \frac{1}{4}F^{ij}F_{ij}\Big). \end{aligned} \tag{1.53}$$

Proceeding with the definition of the canonical momenta associated with the field-independent variables at first order, which are A_0, A_i, and F_{ij}, we obtain:

$$\begin{aligned} \Pi^0(x) &\approx 0, \\ \Pi^i(x) &= -F^{0i} = E^i, \\ \Pi^{ij}(x) &\approx 0. \end{aligned} \tag{1.54}$$

From the above relations, the first and third expressions correspond to primary constraints in the Dirac sense, marked as "weakly zero." The quantity F^{0i} serves as the conjugate momentum of A_i, meaning it is not treated as an independent field from the start. The momenta associated with the particles are:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + eq\vec{A}(\vec{r}), \tag{1.55}$$

$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{R}}} = M\dot{\vec{R}} + gQ\vec{B}(\vec{R}). \tag{1.56}$$

From these expressions, we can solve for the velocities of the particles in terms of the momenta and fields. Now, we express the temporal component of the interaction term with the magnetic monopole as:

$$\begin{aligned} \int d^3x B_0(x) J^0_m(x) &= \int d^3x \int d^4y \left[{}^*F_{0k} f^k(y-x) \right] gQ\delta^{(3)}(\vec{x}-\vec{R}) \\ &= \frac{1}{2} gQ \int d^3y \epsilon_{ijk} F_{ij}(y) f^k(\vec{y}-\vec{R}). \end{aligned} \tag{1.57}$$

This result suggests a convenient definition: $b_{ij} \equiv gQ\epsilon_{ijk} f^k(x-R)$, which allows us to rewrite the last equation as:

$$\int d^3x B_0(x) J^0_m(x) = \frac{1}{2} \int d^3x b_{ij}(x) F_{ij}(x),$$

where b_{ij} depends on the location of the monopole. We can now treat $\Pi_0 = 0$ not as a constraint but rather as a multiplier. That is because there are no temporal derivatives of A_0. Also note that we do not treat F^{0i} as a canonical variable since it turns out to be a conjugate momenta[11]. After performing the Legendre transformation on the fields, the total Hamiltonian can be written as:

$$\begin{aligned} H^* &= \int d^3x \left[\frac{1}{2}\Pi_i^2 - \frac{1}{4}F_{ij}^2 + \frac{1}{2}F_{ij}(f_{ij} + b_{ij}) \right] - \int d^3x A_0 \left[\partial_i \Pi^i - eq\delta^{(3)}(\vec{x} - \vec{r}) \right] \\ &\quad + \int d^3x u_{ij}(x)\Pi^{ij}(x) + \frac{(\vec{p} - eq\vec{A}(\vec{r}))^2}{2m} + \frac{(\vec{P} - gQ\vec{B}(\vec{R}))^2}{2M} \\ &= H + \int d^3x u_{ij}(x)\Pi^{ij}(x). \end{aligned} \tag{1.58}$$

where u_{ij} are Lagrange multipliers. The term H represents the canonical Hamiltonian, while f_{ij} is given by $f_{ij} = \partial_i A_j(x) - \partial_j A_i(x)$. The function A_0 acts as a Lagrange multiplier and reveals a primary constraint in the Hamiltonian formulation, explicitly written as

$$\phi = \left(\partial_i \Pi^i - eq\delta^{(3)}(\vec{x} - \vec{r}) \right) \approx 0, \tag{1.59}$$

which corresponds to the Gauss constraint in the presence of electric charges. This constraint is first-class, meaning its Poisson bracket with the Hamiltonian vanishes. Along with $\Pi^{ij}(x)$ from Eq. (4.18), these represent the system's constraints.

In Eq. (4.22), we observe that the kinetic energy of the particles generates a coupling with the vector potentials, affecting the covariant momenta of the particles. For a charged particle, this coupling naturally emerges from the interaction term in the action, given by

$$\begin{aligned} \int d^3x \left(-A_i(x) J_c^i(x) \right) &= -eq \int d^3x \left(A_i(x) \dot{r}^i(t) \delta^{(3)}(\vec{x} - \vec{r}) \right) \\ &= -eq \int d^3x \left[A_i(x) \dot{r}^i(t) \delta^{(3)}(\vec{x} - \vec{r}) \right] = eq\vec{A} \cdot \dot{\vec{r}}. \end{aligned} \tag{1.60}$$

Once again, considering the interaction between a magnetic monopole and a field, the Lagrangian density leads to the following expression:

$$
\begin{aligned}
-\int d^3x \left(B_i(x) J^i_m(x)\right) &= -\int d^3x \int d^4y \, ({}^*F_{i\mu}(x) f^\mu(y-x) J^\mu_m(x)) \\
&= -gQ \int d^3x \int d^4y \left({}^*F_{i\mu}(x) f^\mu(y-x) \dot{R}^i(t) \delta^{(3)}(\vec{x} - \vec{R}(t))\right) \\
&= -gQ \int d^3x \int d^4y \Big({}^*F_{i0}(x) f^0(y-x) \\
&\quad + {}^*F_{ij}(x) f^j(y-x)\Big) \dot{R}^i(t) \delta^{(3)}(\vec{x} - \vec{R}(t)).
\end{aligned} \tag{4.25}
$$

By utilizing the expression for $B_\mu(x)$ and the corresponding magnetic monopole current, along with the definition ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, we can manipulate equation (4.25) to extract the magnetic "potential" vector:

$$
B_i(\vec{R}(t)) = \int d^3x \, \epsilon_{ijk} F^{0i} f^j(y - \vec{R}(t)) = \int d^3x \, \epsilon_{ijk} \Pi^i f^j(y - \vec{R}(t)), \tag{4.26}
$$

To correctly interpret this result in terms of the "magnetic momentum," one must analyze how the interaction manifests through the potential. This allows for different choices for $f^j(y-x)$. We now establish the standard algebra of Poisson brackets for fields and matter at equal times. The nonzero ones are:

$$
\begin{aligned}
\left\{F_{ij}(\vec{x}), \Pi^{kl}(\vec{y})\right\} &= \frac{1}{2}\left(\delta^k_i \delta^l_j - \delta^l_i \delta^k_j\right) \delta^{(3)}(\vec{x} - \vec{y}), \\
\left\{A_\mu(\vec{x}), \Pi^\nu(\vec{y})\right\} &= \delta^\nu_\mu \delta^{(3)}(\vec{x} - \vec{y}), \\
\left\{r^i, p_j\right\} &= \delta^i_j, \\
\left\{R^i, P_j\right\} &= \delta^i_j.
\end{aligned} \tag{1.61}
$$

Using these brackets, we proceed to ensure the preservation of constraints over time to determine whether secondary constraints arise (or, in some cases, to solve for a Lagrange multiplier). Preserving the primary constraint $\Pi^{ij}(x) \approx 0$, we obtain:

$$\left\{\Pi^{ij}(\vec{x}), H^*\right\} = K_{ij}(x) = F_{ij}(x) - f_{ij}(x) - b_{ij}(x) \approx 0. \tag{1.62}$$

Since the conjugate momentum $\Pi^{ij}(x)$ has a nonzero Poisson bracket only with the last two terms in the first integral of (1.58), where the fields F_{ij} are present. Equation (1.62) represents a new constraint. We proceed with its time preservation, obtaining:

$$\begin{aligned} \dot{K}_{ij} &= \{K_{ij}, H^*\} \approx 0 \Rightarrow \\ &- 2u_{ij}(x) + \partial_i \Pi^j(x) - \partial_j \Pi^i(x) - gQ\epsilon_{ijk}\left(P_l + gQB_l(\vec{R})\right)\frac{\partial f^k(\vec{x}-\vec{R})}{\partial R^l} = 0, \end{aligned} \tag{1.63}$$

from which we can solve for the Lagrange multipliers, obtaining

$$u_{ij}(x) = -\frac{1}{2}\left(\partial_j F_{0i}(x) - \partial_i F_{0j}(x)\right) + gQ\epsilon_{ijk}\left(P_l + gQB_l(\vec{R})\right)\frac{\partial f^k(\vec{x}-\vec{R})}{\partial R^l}. \tag{1.64}$$

These expressions must then be substituted back into the Hamiltonian (1.58). Thus, we have the constraints: $\Pi^{ij} \approx 0$, $\phi = \left(\partial_i \Pi^i - eq\delta^{(3)}(\vec{x}-\vec{r})\right) \approx 0$ and $K_{ij} = F_{ij} - f_{ij} - b_{ij} \approx 0$. From this, we can see that Π^0 and the Gauss constraint are first-class, whereas the others are second-class constraints. Consequently, following Dirac's procedure, we introduce Dirac brackets to obtain a consistent quantum theory, allowing us to strongly impose the second-class constraints to zero. This approach allows us to bypass the cumbersome process of explicitly constructing the Dirac matrix and inverting it, both of which are required to obtain the desired brackets. Instead, we express the field F_{ij} and its conjugate momentum Π^{ij} in terms of the remaining canonical variables. We achieve this by strongly enforcing the second-class constraints and substituting everything back into the Hamiltonian.

At that stage, the only brackets we need to consider are the Dirac brackets, which in this case coincide with the Poisson brackets. These apply only to the remaining canonical variables: the fields A_0, A_i, and their conjugate momenta Π^0 and Π^i. This is because we have already eliminated F_{ij} and Π^{ij}, as they were involved in the second-class constraints.

Our Hamiltonian is given by equation (1.58), where we first incorporated the Lagrange multipliers u_{ij} that were previously solved for. Then, by strongly enforcing $\Pi^{ij} = 0$, we eliminate this variable completely [12]. Substituting $F_{ij}(\vec{x}) = f_{ij}(\vec{x}) + b_{ij}(\vec{x})$, we obtain:

$$H^* = \int d^3x \left[\frac{1}{2}\Pi_i^2 + \frac{1}{4}(f_{ij} + b_{ij})^2 - A_0 \left(\partial_i \Pi^i - eq\delta^{(3)}(\vec{x} - \vec{r}) \right) \right] + \frac{(\vec{p} - eq\vec{A}(\vec{r}))^2}{2m} + \frac{(\vec{P} - gQ\vec{B}(\vec{R}))^2}{2M}. \tag{1.65}$$

To simplify the formulation further [6], we introduce an additional constraint in an *ad hoc* manner. This transformation converts a set of previously first-class constraints into second-class ones. For instance, by choosing the temporal gauge $A_0 = 0$ and treating this as a new constraint, we obtain a new pair of second-class constraints: Π^0 and A_0. Following the earlier procedure, we eliminate these variables from the formalism, leaving only the fields A_i and their conjugate momenta

Π^i, along with the algebra generated by their Dirac brackets, which in this case coincide with the Poisson brackets.

Finally, we proceed with quantization using the standard approach. To clarify the process, we recall that Dirac brackets are replaced by commutators, and canonical variables become operators. These operators must act within the physical sector of the theory. The physical states $|\Psi\rangle_{\text{phy}}$ must satisfy the following constraint, which generalizes Gauss's law:

$$\left(\partial_i \hat{\Pi}^i - eq\delta^{(3)}(\vec{x}-\vec{r})\right)|\Psi\rangle_{\text{phy}} = 0. \tag{1.66}$$

From the system's structure, we obtain the fundamental commutators that do not vanish:

$$\begin{aligned} \left[\hat{A}_i(x), \hat{\Pi}^j(y)\right] &= i\delta_i^j \delta^{(3)}(\vec{x}-\vec{y}), \\ \left[\hat{r}^i, \hat{p}_j\right] &= i\delta_j^i, \\ \left[\hat{R}^i, \hat{P}_j\right] &= i\delta_j^i. \end{aligned} \tag{1.67}$$

The dynamics governing the evolution of physical states is determined by the Schrödinger equation, which explicitly takes the form:

$$\begin{aligned} &\Bigg[\int d^3x \left(\frac{1}{2}\hat{\Pi}_i^2 + \frac{1}{4}(\hat{f}_{ij} + b_{ij})^2\right) \\ &\qquad + \frac{(\hat{\vec{p}} - eq\hat{\vec{A}})^2}{2m} + \frac{(\hat{\vec{P}} - gQ\hat{\vec{B}})^2}{2M}\Bigg]|\Psi\rangle_{\text{fis}} = i\frac{\partial}{\partial t}|\Psi\rangle_{\text{fis}}. \end{aligned} \tag{1.68}$$

It is worth emphasizing that, in the absence of a magnetic monopole, the Hamiltonian simplifies to the standard Maxwell theory coupled to electric charges. In such a case, the magnetic field is solely determined by the exterior derivative of the potential:

$$F_{ij} = \partial_i A_j - \partial_j A_i.$$

However, when a magnetic monopole is present, an additional contribution appears, which manifests as an external field term b_{ij}. Now our next step is to use the geometric representation explained in the following chapter, so that we can express the operators using geometric objects with the constraint that they obey the commutator relations described above. In the next chapter the geometric representation will be explained and later on it will be used in the case fo the dynamic dyone.

Chapter 2

Geometrical representation

Once our theory is quantized we want to obtain a representation for the operators in a geometric language. As we will see, we want a realization involving paths, loops and surfaces such that it satisfies the commutators algebras and solves the constraints. We will use this geometric base such that the wave functional now depends on this objects. We first need to introduce the path and surface groups with their respective derivatives and form factors which will allow us to choose correctly the operators of our field theory. [6][9][7].

2.1 Path and Surface groups

The geometric formulation employs operators that act on curves, specifically 1-dimensional surfaces. We define the space of paths P as the collection of all

1-surfaces on a manifold M. More precisely, these are maps $\gamma : [s_1, s_2] \cup \cdots \cup [s_{n-1}, s_n] \to M$ that are smooth within each interval $[s_i, s_{i+1}]$ and continuous across the entire domain. The composition of two paths is given by:

$$\gamma_1 \circ \gamma_2 = \begin{cases} \gamma_1(2t) & t \in \left[0, \frac{1}{2}\right] \\ \gamma_2\left(2(t - \frac{1}{2})\right) & t \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{2.1}$$

In simple terms, this operation corresponds to traversing the path γ_1 first, followed by γ_2. Our primary focus is on the geometric properties of the paths rather than their specific parametrization. The composition of paths does not have an abelian group structure, for that reason, it is natural to introduce a equivalence relation between curves. We define the **Form Factor** as follows:

$$T^i(\vec{x}, \gamma) = \int_\gamma dy^i \, \delta^n(\vec{x} - \vec{y}). \tag{2.2}$$

We can simply think of the form factor as a function that extracts the tangent vector at a given point. For that reason it is easy to note that the form factor is independent of the parametrization of the path γ. For that reason we it defines an equivalence relation between curves,

$$\gamma_1 \sim \gamma_2 \quad \Longleftrightarrow \quad T^i(\vec{x}, \gamma_1) = T^i(\vec{x}, \gamma_2)$$

Now all parametrization of a path belong to a single equivalence class. Because the composition of two paths is another path, the composition is associative, we can define the identity path such that $id \circ \gamma = \gamma \circ id = \gamma$, and finally we define the inverse path γ^{-1} as the path traversed in the opposite direction as γ. This shows that the set of paths forms a group under composition.

Having defined the form factor, we now explore how it naturally leads to the structure of an Abelian group, known as the **Abelian Path Space**. The key observation is that the form factor $T^i(\vec{x}, \gamma)$ assigns a field contribution based on the integration over the path γ. It is easy to notice that:

$$T^i(\vec{x}, \gamma_1 \circ \gamma_2) = T^i(\vec{x}, \gamma_1) + T^i(\vec{x}, \gamma_2), \tag{2.3}$$

which suggests that the space of paths inherits a group-like structure. Since the form factor depends only on the total integral over the path, the order of composition does not affect the result:

$$\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1.$$

This last property ensures that the set of equivalence classes of paths, under this composition rule, forms an **Abelian group**. Thus, the space of paths modulo this equivalence relation is known as the **Abelian Path Space**.

We can generalize this notion to p-surfaces. In this generalized setting, the fundamental objects are p-dimensional surfaces embedded in a manifold M. We define the space of oriented p-surfaces, denoted S_p, as the collection of maps

$$\Sigma : [s_1, s_2] \times \cdots \times [s_1^p, s_2^p] \cup \cdots \cup [s_{n-1}^1, s_n^1] \times \cdots \times [s_{n-1}^p, s_n^p] \to M$$

that are smooth within each subinterval $[s_i, s_{i+1}]$ and globally continuous across the domain. The composition of these surfaces follows a similar prescription to that of paths, as outlined in Equation (2.1). However, since our focus is on the geometric properties of these objects rather than their specific parameterizations, we introduce equivalence classes using the form factor again. This ensures that two different parametrizations of the same p-surface are treated as equivalent. To make this an abelian group,

$$T^{i_1 \dots i_p}(\vec{x}, \Sigma) = \int_\Sigma d\Sigma^{i_1 \dots i_p}(\vec{y})\, \delta^{D-1}(\vec{x} - \vec{y}).$$

Two p-surfaces are then considered equivalent if they have the same form factors. We are interested in the **group of surfaces**, that is $p = 2$. where the form factor is wirtten as:

$$T^{ij}(\vec{x}, \Sigma) = \int_\Sigma d\Sigma^{ij}_y\, \delta^{(3)}(\vec{x} - \vec{y}). \tag{2.4}$$

where

$$d\Sigma^{ij}_y = \frac{\partial y^i}{\partial s}\frac{\partial y^j}{\partial t} - \frac{\partial y^i}{\partial t}\frac{\partial y^j}{\partial s}. \tag{2.5}$$

Here t and s are the parameters needed to parametrize the surface. In the same way as paths, two surfaces Σ_1 and Σ_2 are equivalent if they have the same form factor, that is:

$$\Sigma_1 \sim \Sigma_2 \quad \Longleftrightarrow \quad T^{ij}(\vec{x}, \Sigma_1) = T^{ij}(\vec{x}, \Sigma_2).$$

We now have constructed the **group of surfaces**. It is important to note that for both the path-space and surface-space there are important subgroups, that is the **group of loops** and the groups of **closed surfaces** respectively. This groups will also play a crucial role in the next chapters. Now we need to define the derivatives associated with each of this spaces that will allow us to represent our operators using them.

2.2 Path and loop derivatives

Remember we are now working with wave functionals, where the wave function ψ now depends on a path γ (or surfaces Σ as we will see next). We can chose the base $|\gamma\rangle$ from which we can construct this representation of the wave function, that is: $\langle\gamma \mid \psi\rangle = \psi(\gamma)$. Now we can define the **path derivative** as follows: We consider an infinitesimal path, or "increment", denoted as $u_{\vec{x}}$, which starts at the point $\vec{x}$ and extends slightly in the direction of the tangent vector $\vec{u}$. This can be written as $u_{\vec{x}}^{\vec{x}+\vec{u}}$. The functional $\psi(\gamma)$, when evaluated on the slightly perturbed path (up to first order in $\vec{u}$), takes the form

$$\psi(\gamma \circ u_{\vec{x}}) = \psi(\gamma) + u^i \delta_i(\vec{x})\psi(\gamma).$$

and thus we have:

$$u^i \delta_i(\vec{x})\psi(\gamma) = \psi(\gamma \circ u_{\vec{x}}) - \psi(\gamma) \tag{2.6}$$

In similar manner, we now define the **loop derivative**. Given a path, we define an "infinitesimal cycle" as the increment δC, which takes the form

$$\delta C = u_{\vec{x}} \circ v_{\vec{x}+\vec{u}} \circ \bar{u}_{\vec{x}+\vec{u}+\vec{v}} \circ \bar{v}_{\vec{x}+\vec{v}}$$

representing the boundary of an infinitesimal surface element σ, whose area element is given by

$$\sigma^{ij} = u^i v^j - u^j v^i,$$

generated by the vectors $\vec{u}, \vec{v}$. From this, we naturally arrive at the definition of the **loop derivative**:

$$\psi(\gamma \circ \delta C) - \psi(\gamma) = \frac{1}{2}\sigma^{ij}\Delta_{ij}(\vec{x})\psi(\gamma) \tag{2.7}$$

which holds to first order in σ. It can be shown that applying the definition of path differentiation (Eq. (2.6)) to the infinitesimal cycle δC and substituting into Eq. (2.7), we obtain directly:

$$\Delta_{ij}(\vec{x}) = \partial_i \delta_j(\vec{x}) - \partial_j \delta_i(\vec{x}). \tag{2.8}$$

We will not go deeper into the definition of these objects, as they are already well studied [13]. We know write some useful relations between the path and loop derivatives with the form factor. Note that:

$$T^i(\vec{x}, \gamma \circ u_y) = \int_{\gamma \circ u_y} dy^i\, \delta^n(\vec{x} - \vec{y}) = T^i(\vec{x}, \gamma) + u^j \delta^i_j \delta^n(\vec{x} - \vec{y}),$$

which leads to the result

$$\delta_i(\vec{y}) T^j(\vec{x}, \gamma) = \delta^j_i \delta^n(\vec{x} - \vec{y}). \tag{2.9}$$

Similarly, for the loop derivative, we obtain:

$$\Delta_{ij}(\vec{y})T^k(\vec{x},\gamma) = \delta^k_j \partial_i \delta^n(\vec{x}-\vec{y}) - \delta^k_i \partial_j \delta^n(\vec{x}-\vec{y}). \tag{2.10}$$

We now analyze the divergence of the form factor in the context of paths. As an example of a calculation involving this factor, we obtain:

$$\partial_i T^i(\vec{x},\gamma) = -\int dy^i \partial^y_i \delta^{(3)}(\vec{x}-\vec{y}) = -\varrho(\vec{x},\gamma). \tag{2.11}$$

In the previous expression, we have defined $\varrho(\vec{x},\gamma)$ as the **distribution of path endpoints** γ, given by:

$$\varrho(\vec{x},\gamma) = \sum_s \left(\delta^{(3)}(\vec{x}-\vec{\beta}_s) - \delta^{(3)}(\vec{x}-\vec{\alpha}_s)\right). \tag{2.12}$$

This distribution is supported at the endpoints or boundaries of the path segments ($\partial\gamma$). The sign distinguishes whether the points correspond to the starting positions ($\vec{\alpha}_s$) or the final positions ($\vec{\beta}_s$) of these segments.

Notice that these relations are important as they will make the operators obey the commutator algebra.

2.3 Open p-surface and closed p-surface derivatives

We can now define the extended derivatives for p-surfaces [13], the open p-surface derivative $\delta_{i_1 \ldots i_p}(\vec{x})$ and the closed p-surface derivative $\Delta_{i_1 \ldots i_{p+1}}(\vec{x})$. Their definitions follow a structure analogous to the one-dimensional case:

$$\sigma^{i_1 \ldots i_p} \delta_{i_1 \ldots i_p}(\vec{x}) \psi(\Sigma) = \psi(\Sigma \circ \sigma_x) - \psi(\Sigma), \tag{2.13}$$

$$\frac{1}{(p+1)!} \sigma^{i_1 \ldots i_{p+1}} \Delta_{i_1 \ldots i_{p+1}}(\vec{x}) \psi(\Sigma) = \psi(\Sigma \circ \delta\sigma) - \psi(\Sigma). \tag{2.14}$$

As before there is a relation between both derivatives

$$\Delta_{i i_1 \ldots i_{p+1}}(\vec{x}) = \frac{1}{p!} \partial_{[i} \delta_{i_1 \ldots i_p]}. \tag{2.15}$$

Now for our interest we study the case for $p = 2$, that is the group of surfaces. We are interested in the actions of the open and closed surface derivatives on the form factor. The **surface derivative**, denoted as $\delta_{ij}(x)$. This derivative measures the response of surface-dependent functions $\Psi(\Sigma)$ when an infinitesimal surface element, with area σ_{ij}, is "attached" to its argument Σ at the point x. To first order in σ_{ij}, we have:

$$\Psi(\delta\Sigma \circ \Sigma) - \Psi(\Sigma) = \sigma^{ij} \delta_{ij}(x) \Psi(\Sigma), \tag{2.16}$$

where

$$\sigma^{ij} = u^i v^j - v^j u^i, \tag{2.17}$$

is the surface element generated by the vectors $\vec{u}$ and $\vec{v}$. The surface derivative $\delta_{ij}(x)$, which generalizes the path derivative for open surfaces, should not be confused with the loop derivative $\triangle_{ij}(x)$. By using $\delta_{ij}(x)$, we can define the closed surface derivative (a three-index derivative) $\triangle_{ijk}$ through the relation:

$$\Psi(\delta\Sigma \circ \Sigma) - \Psi(\Sigma) = V^{ijk} \triangle_{ijk}(x)\Psi(\Sigma). \tag{2.18}$$

Just as the loop derivative (which involved two indices and attached infinitesimal closed paths) was derived from the path derivative (which involved a single index), the closed surface derivative introduces an infinitesimal "cube" (volume element) V^{ijk} into the argument of $\Psi(\Sigma)$. The infinitesimal volume can be expressed as:

$$V^{ijk} = u^{[i} v^j w^{k]}, \tag{2.19}$$

where $\vec{u}$, $\vec{v}$, and $\vec{w}$ are the infinitesimal vectors generating this volume. The relationship between the open and closed surface derivatives follows directly from their definition and is given by:

$$\triangle_{ijk}(x) = \partial_i \delta_{jk}(x) + \partial_j \delta_{ki}(x) + \partial_k \delta_{ij}(x). \tag{2.20}$$

The surface derivative acting on the form factor yields the following:

$$\delta_{ij}(\vec{x}) T^{kl}(\vec{y}, \Sigma) = \frac{1}{2} \left(\delta_i^k \delta_j^l - \delta_i^l \delta_j^k \right) \delta^{(3)}(\vec{x} - \vec{y}). \tag{2.21}$$

Another useful result is to calculate the divergence of the form factor. In the path formulation when computing the divergence of the form factor the result describes a distribution of points corresponding to the start and endpoints of path segments, namely, the boundary of the paths $\partial\gamma$. Similarly, when calculating the divergence of our form factor in surface space we obtain:

$$\partial_i T^{ij}(\vec{x}, \Sigma) = T^j(\vec{x}, \partial\Sigma). \tag{2.22}$$

Where $\partial\Sigma$ is the border of the surface Σ. So, the divergence is the form factor evaluated in the path $\partial\Sigma$ that limits the surface. We now have all the tools needed for using the geometric representation. For more detail mathematics, the following source is recommended: [13].

2.4 Geometric formulation of Schwinger action

We will now proceed with the example presented in Chapter 1. With the simplification that we will assume we have static magnetic and electric charges. That is:

$$J_e^i = 0, \qquad J_m^i = 0$$

$$J_e^0 = eq\delta^3(x), \qquad J_m^0 = gQ\delta^3(x)$$

considering that the charges are on the origin of our reference frame. This simplifies things a bit, for instance, we have that the first-class constraint now becomes:

$$\left(\partial_i \hat{\Pi}^i - eq\delta^3(x)\right)|\Psi\rangle = 0 \tag{2.23}$$

and the magnetic monopole term $b_{ij}(x)$ now becomes:

$$b_{ij}(x) = gQ\epsilon_{ijk} f^k(x) \tag{2.24}$$

and finally the Hamiltonian becomes

$$\hat{H} = \int d^3x \left[\frac{1}{2}\hat{\Pi}_i^2 + \frac{1}{4}(\hat{f}_{ij} \; + \; \hat{b}_{ij})^2 \right]. \tag{2.25}$$

Let us now perform the following transformation on the fields, which is convenient for the analysis and will serve us for the next chapter in the dynamic case. We define a "new" electric field by subtracting from the "original" electric field a constant term that represents the field of a point charge:

$$\pi^i(\vec{x}) = \Pi^i(\vec{x}) - E^i_{\text{point charge}}(x), \tag{2.26}$$

where

$$E^i_{\text{point charge}}(x) = \frac{eq}{4\pi} \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|^3}, \tag{2.27}$$

so that Gauss's constraint is now written as

$$\left(\partial_i \hat{\pi}^i(x) + \partial_i \hat{E}^i_{\text{point charge}}(x) - eq\delta(x) \right) |\Psi\rangle_{\text{phys}} = 0 \Rightarrow \partial_i \hat{\pi}^i(x) = 0, \tag{2.28}$$

due to the fact that

$$\partial_i \hat{E}^i_{\text{point charge}}(x) = \frac{eq}{4\pi} \partial_i \left[\frac{(x - x_0)^i}{|\vec{x} - \vec{x}_0|^3} \right] = -\frac{eq}{4\pi} \partial_i \partial_i \left[\frac{1}{|\vec{x} - \vec{x}_0|} \right]$$

$$= -\frac{eq}{4\pi}\nabla^2\left[\frac{1}{|\vec{x}-\vec{x}_0|}\right] = eq\delta^3(\vec{x}-\vec{x}_0).$$

A more refined formulation of the previous discussion can be considered,we will call this scheme the “bundles” scheme. To achieve this, we introduce a transformed field given by:

$$\hat{\pi}^i(\vec{x}) = \hat{\Pi}^i(\vec{x}) - \hat{E}^i_{\text{point charge}}(x) = \hat{\Pi}^i(\vec{x}) + eqT^i(\vec{x},\gamma^{\vec{x}_0}) \tag{2.29}$$

where $T^i(\vec{x},\gamma^{\vec{x}_0})$ characterizes the distribution of tangents of the parallel ”bundles” that originate at spatial infinity and converge at the position $\vec{x}_0$. Once again, it can be directly verified that the constraint condition still holds, now as a consequence of:

$$\partial_i\hat{E}^i_{\text{point charge}}(x) = -eq\partial_iT^i(\vec{x},\gamma^{\vec{x}_0}) = eq\delta^3(\vec{x}-\vec{x}_0). \tag{2.30}$$

From this, we define the action of the modified electric field operator, which incorporates this shift, as well as the potential acting on the path-dependent functionals as:

$$\hat{\pi}^i(\vec{x})\Psi(\gamma) = eq\,T^i(\vec{x},\gamma)\Psi(\gamma), \tag{2.31}$$

$$\hat{A}_i(\vec{x})\Psi(\gamma) = \frac{i}{eq}\delta_i(\vec{x})\Psi(\gamma). \tag{2.32}$$

It can easily be shown that this operators obey the commutator algebra (1.67). Now, notice that,

$$\begin{aligned}\hat{f}_{ij} &= \partial_i \hat{A}_j - \partial_j \hat{A}_i \\ &= \frac{1}{eq}(\partial_i \delta_j - \partial_j \delta_i) \\ &= \frac{1}{eq} \triangle_{ij}\end{aligned}$$

Thus, we can write the Hamiltonian in a completely geometric and consistent manner:

$$\hat{H} = \int d^3x \left[\frac{eq}{2} \left(T^i(\vec{x}, \gamma) - T^i(\vec{x}, \gamma^{\vec{x}_0}) \right)^2 - \frac{1}{4(eq)^2} \left(\Delta_{ij}(\vec{x}) - ieq\, b_{ij} \right)^2 \right], \qquad (2.33)$$

recalling that $\vec{x}_0 = \vec{0}$. Finally, the constraint is implemented as

$$\partial_i \hat{\Pi}^i(x) \, |\Psi\rangle_{phy} = 0 \quad \longrightarrow \quad \partial_i T^i(\vec{x}, \gamma) \, |\Psi\rangle_{phy} = 0,$$

Chapter 3

Dynamic Dyons

3.1 First order Schwinger action with electric and magnetic "charged" particles

We now want to study the Schwinger action at first order in the presence of a new type of particle that possesses both electric and magnetic "charge". We will call this particle: **Dyon**.The following was mainly built on the work of Ernesto Fuenmayor and his previous studies [12, 14, 15]. Consider the action similar to the one studied as an example before:

$$\begin{aligned} S = \int d^4x \Big(- (A_\mu + B_\mu)\, J^\mu_D - \frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \Big) \\ + \int dt \left(\frac{1}{2} m \dot{\vec{r}}^2 \right) \end{aligned} \tag{3.1}$$

Once again, our dynamical variables are given by the independent fields A_μ and $F^{\mu\nu}$, respectively, along with the position coordinates $\vec{r}$ of particles carrying both electric and magnetic charges. Additionally, the current associated with the dyone is represented by J^μ_D, which is expressed as

$$J^\mu_D = eg\, v^\mu(t)\delta^{(3)}(\vec{x}-\vec{r}).$$

As in section 1 following the Hamiltonian formulation, we define the conjugate momenta and decompose the action in $3+1$ dimensions obtaining very similar as before:

$$\begin{aligned} H^* =& \int d^3x \left[\frac{1}{2}\Pi_i^2 - \frac{1}{4}F_{ij}^2 + \frac{1}{2}F_{ij}(f_{ij}+b_{ij})\right] - \int d^3x A_0 \left[\partial_i \Pi^i - eq\delta^{(3)}(\vec{x}-\vec{r})\right] \\ &+ \frac{1}{2m}\left[\vec{p} - eg\big(\vec{A}(\vec{r}) + \vec{B}(\vec{r})\big)\right]^2 + \int d^3x\, u_{ij}(x)\Pi^{ij}(x) \\ =& H + \int d^3x\, u_{ij}(x)\Pi^{ij}(x). \end{aligned} \tag{3.2}$$

where u_{ij} are the Lagrange multipliers, and H represents the canonical Hamiltonian. As before, e (g) represents the elemental electric (magnetic) charge, and q is a number that specifies the total charge of the particle. The definitions are given by

$$f_{ij} = \partial_i A_j(x) - \partial_j A_i(x), \quad b_{ij}(x) \equiv eg\epsilon_{ijk}f^k(\vec{x}-\vec{r}). \tag{3.3}$$

We have the freedom to choose f^k, as discussed in section 1. The function A_0 serves as a Lagrange multiplier, making it straightforward to identify the constraint

$$\phi = \big(\partial_i \Pi^i - eg\delta^{(3)}(\vec{x}-\vec{r})\big) \approx 0, \tag{3.4}$$

which corresponds to the Gauss constraint in the presence of electric charges. Along with

$$\Pi^{ij}(x) = 0, \tag{3.5}$$

these constraints define the system's fundamental restrictions. By implementing canonical quantization, we obtain the following result. After resolving the second-class constraints appearing in the canonical formalism—using Dirac bracket constructions—we obtain canonical variable pairs (operators) associated with the fields

$$\hat{A}_i(x), \quad \hat{\Pi}^i(x), \tag{3.6}$$

as well as the variables corresponding to the particles

$$\hat{r}^i, \quad \hat{p}_j, \tag{3.7}$$

which satisfy the expected algebraic structure. The only commutators that do not vanish, inherited from the system, are given by:

$$\left[\hat{A}_i(x), \hat{\Pi}^j(y)\right] = i\delta_i^j \delta^{(3)}(\vec{x} - \vec{y}), \tag{3.8}$$

$$\left[\hat{r}^i, \hat{p}_j\right] = i\delta_j^i. \tag{3.9}$$

These operators must be applied in the physical sector of the states $|\Psi\rangle_{\text{phy}}$, within a space that satisfies the generalized Gauss's law. It is in this context that the presence of the charge associated with the Dyons becomes evident:

$$\left(\partial_i \hat{\Pi}^i - eg\delta^{(3)}(\vec{x} - \vec{r})\right) |\Psi\rangle_{\text{phy}} = 0. \tag{3.10}$$

Finally, the time evolution of the physical states is governed by the Schrödinger equation, which is expressed as:

$$\left[\int d^3x \left(\frac{1}{2}\hat{\Pi}_i^2 + \frac{1}{4}(\hat{f}_{ij} + b_{ij})^2\right) + \frac{(\hat{\vec{p}} - eg(\hat{\vec{A}}(\vec{r}) + \hat{\vec{B}}(\vec{r})))^2}{2m}\right] |\Psi\rangle_{\text{phy}} \\ = i\frac{\partial}{\partial t}|\Psi\rangle_{\text{phy}}. \tag{3.11}$$

Here, the canonical Hamiltonian becomes evident, and we can finally put to use the geometric formulation

3.2 Geometric formulation for dynamic dyons

There are two ways to use the geometric formulation for the operators concerning the dynamic dyons, the **direct** way or the **dual** way [15][6][9][7]. We will use the latter, as it will allow us to give a geometric interpretation of the constraint (3.10). Given the properties of the form factor and the surface derivatives we propose the following operators acting on the state $\Psi(\Sigma)$:

$$\hat{A}_i(\vec{x})\Psi(\Sigma) = eg\,\epsilon_{ijk}T^{jk}(\vec{x},\Sigma)\Psi(\Sigma), \tag{3.12}$$

$$\hat{\pi}^i(\vec{x})\Psi(\Sigma) = -\frac{i}{eg}\epsilon^{ijk}\delta_{jk}(\vec{x})\Psi(\Sigma), \tag{3.13}$$

where $\hat{\pi}^i$ is defined the same way as in eq. (2.29). We have already shown that the operators satisfy the algebraic structure. The Hamiltonian can be written as in the last chapter 2, that is:

$$\hat{H} = \int d^3x \left[\frac{1}{2} \left(\hat{\pi}^i(\vec{x}) + \hat{E}^i_{\text{point charge}}(x) \right)^2 + \frac{1}{4} (\hat{f}_{ij} \; + \; \hat{b}_{ij})^2 \right] + \frac{(\hat{\vec{p}} - eg(\hat{\vec{A}}(\vec{r}) + \hat{\vec{B}}(\vec{r})))^2}{2m} \tag{3.14}$$

Now, we will consider the following calculations

$$\epsilon^{mij} \partial_i A_j = \epsilon^{mij} \partial_i \epsilon_{jkl} T^{kl} = \left(\delta^m_k \delta^i_l - \delta^m_l \delta^i_k \right) \partial_i T^{kl} = \partial_i T^{mi} - \partial_i T^{im} = 2 \partial_i T^{mi} \tag{3.15}$$

Remember the divergence of the form factor of the surfaces relates to the boundary path of the surface, we use the identity $\partial_i T^{ij}(\vec{x}, \Sigma) = T^j(\vec{x}, \partial\Sigma)$, leading to the following relations:

$$\epsilon^{mij} \partial_i A_j = 2 \partial_i T^{mi}(\Sigma) = 2 T^m(\partial\Sigma), \tag{3.16}$$

$$\epsilon^{mij} \partial_j A_i = -2 \partial_i T^{mi}(\Sigma) = -2 T^m(\partial\Sigma). \tag{3.17}$$

From this relations we can write:

$$\hat{f}_{ij} = \frac{2}{3} \epsilon_{mij} T^m(\partial\Sigma) \tag{3.18}$$

and if we define

$$^*\hat{b}^i = \frac{1}{2} \epsilon^{ikl} \hat{b}_{kl} \tag{3.19}$$

then it can be easily checked that the Hamiltonian without the dynamic part (lets call it H') can be written as:

$$\hat{H}' = \int d^3x \left[\frac{1}{2} \left(-\frac{i}{eg} \epsilon^{ijk} \delta_{jk}(\vec{x}) \right)^2 + \frac{1}{6^2} \left(2eg \; \epsilon_{mij} \, T^m(\partial\Sigma) + \epsilon_{mij} {}^*b^m \right)^2 \right]. \tag{3.20}$$

Now the generalized Gauss constraint (3.10) transforms into:

$$\left(\frac{i}{eg}\epsilon^{ijk}\partial_i\delta_{jk}(x) + eg\delta^{(3)}(\vec{x}-\hat{\vec{r}})\right)|\Psi\rangle_{phy} = 0 \tag{3.21}$$

where the same interpretations hold for the constants that define the scales of electric and magnetic flux, e and g. In this case, the dyon charge is given by $\tilde{Q}_D = eg$, since we have chosen $q = Q = 1$ for simplicity.

To solve equation (3.21), we assume, without loss of generality, that the wave functionals in the dual surface space take the form:

$$\Psi_D(\Sigma) \equiv \exp\left(i\chi(\Sigma)\right)\Phi_D(\Sigma), \tag{3.22}$$

and now we substitute equation (3.22) into the constraint to determine the functions $\chi(\Sigma)$ and $\Phi(\Sigma)$.

$$\frac{i}{eg}\epsilon^{ijk}\partial_i\delta_{jk}(x)\left[\exp\left(i\chi(\Sigma)\right)\Phi_D(\Sigma)\right] + eg\delta^{(3)}(\vec{x}-\hat{r})\Psi_D(\Sigma) = 0$$

Expanding the derivative inside the first term, we obtain

$$\frac{i}{eg}\epsilon^{ijk}\left[i\partial_i\delta_{jk}(x)\chi(\Sigma)\exp\left(i\chi(\Sigma)\right)\Phi_D + \exp\left(i\chi(\Sigma)\right)\partial_i\delta_{jk}(x)\Phi_D\right] + eg\delta^{(3)}(\vec{x}-\hat{r})\Psi_D = 0$$

For this equation to hold, the function $\chi(\Sigma)$ must satisfy

$$\epsilon^{ijk}\partial_i\delta_{jk}(x)\chi(\Sigma) = (eg)^2\delta^{(3)}(\vec{x}-\hat{\vec{r}}) \tag{3.23}$$

and also:

$$\Phi_D(\Sigma) = \Phi_D(\partial\Sigma)$$

that way $\delta_{jk}(x)\Phi_D = 0$ and with the condition on $\chi(\Sigma)$ outlined above, the constraint holds. A valid choice for $\chi(\Sigma)$ that satisfies equation (3.23) is

$$\chi(\Sigma) = \frac{(eg)^2}{4\pi}\Omega_g(\Sigma), \tag{3.24}$$

where $\Omega_g(\Sigma)$ represents a geometric term associated with the surface. More specifically, it is the solid angle subtended by any surface bounded by the curve $\gamma = \partial\Sigma$, and measured from the position $\vec{r}$, where the magnetic charge of the particle (dyon) is located. We chose $\Omega_g(\Sigma)$ to be:

$$\Omega_g(\Sigma) = \frac{4\pi}{3!\ eg}\int d\Sigma^{ij}_{\vec{y}}\ b_{ij}(\vec{y})$$

that is because:

$$\begin{aligned}
\epsilon^{ijk}\partial_i\delta_{jk}(x)\chi(\Sigma) &= \frac{eg}{3!}\epsilon^{ijk}\partial_i\delta_{jk}(x)\int d\Sigma^{lm}_{\vec{y}}\, b_{lm}(\vec{y}) \\
&= \frac{eg}{3!}\epsilon^{ijk}\partial_i\delta_{jk}(x)\int d\Sigma^{lm}_{\vec{y}}\,(eg)\epsilon_{lms}\ f^s(\vec{y}-\vec{r}) \\
&= \frac{(eg)^2}{3!}\epsilon^{ijk}\epsilon_{lms}\ \delta_{jk}(x)\int d\Sigma^{lm}_{\vec{y}}\ \partial_i f^s(\vec{y}-\vec{r}) \\
&= \frac{(eg)^2}{3!}\epsilon^{ijk}\epsilon_{lms}\ \delta_{jk}(x)\ T^{lm}\,(\vec{r},\Sigma) \\
&= \frac{(eg)^2}{3!}\epsilon^{ijk}\epsilon_{lms}\frac{1}{2}(\delta^l_j\delta^m_k - \delta^m_j\delta^l_k)\ \delta^{(3)}(\vec{x}-\vec{r}) \\
&= (eg)^2\delta^{(3)}(\vec{x}-\hat{\vec{r}})
\end{aligned}$$

Under this choice, the functionals take the final form

$$\Psi_D(\Sigma) \equiv \exp\left(i\frac{(eg)^2}{4\pi}\Omega_g(\Sigma)\right)\Phi_D(\partial\Sigma) \tag{3.25}$$

Notice how now our wave functional is only dependent on a phase which has a geometric connection to magnetic charge, we could say that is a magnetic functional. In our current formulation, the position of the magnetic charge coincides with that of the electric charge (since we consider dyons as sources), so writing the functional (3.25) with a phase depending on the solid angle measured from the electric charge should be equivalent to writing it with a phase depending on the solid angle measured from the magnetic monopole, thus obtaining our electric functional.

Consider an electric charge associated with a particle (such as a dyon) positioned at $\vec{r}$. In this case, we can describe its electric field using the conventional expression:

$$E_e^i = \frac{e}{4\pi\epsilon_0}\frac{(x-r)^i}{|\vec{x}-\vec{r}|^3}, \tag{3.26}$$

Alternatively, we may adopt the "bundle" framework to represent the field. In this approach, the final location of the electric charge at $\vec{r}$ is described using the form factor:

$$E_e^i = -\frac{e}{\epsilon_0}T^i(\vec{x},\gamma^{\vec{r}}) \tag{3.27}$$

As previously discussed, both representations satisfy the necessary divergence condition $\partial_i E_e^i$, apart from constant factors. Moreover, we make use of the relation

${}^*E_{ij} \equiv \epsilon_{ijk} E_e^k$. Consequently, we define:

$$\Omega_e(\Sigma) = \frac{4\pi\epsilon_0}{3!e} \int d^3x \; T^{ij}(\vec{x}, \Sigma) \; {}^*E_{ij}(\vec{x}) \tag{3.28}$$

where $\Omega_e(\Sigma)$ represents the solid angle subtended by the boundary curve of the surface Σ, measured from the position of the electric charge. Notably, this position coincides with that of the magnetic charge. Notice it works because it can be easily shown that:

$$\delta_{jk}(x)\Omega(\Sigma) = -\frac{4\pi}{3!}\epsilon_{jkl}T^l(\vec{x}, \gamma^{\vec{r}})$$

and thus,

$$\epsilon^{ijk}\partial_i\delta_{jk}(x)\chi(\Sigma) = (eg)^2\delta^{(3)}(\vec{x} - \hat{\vec{r}})$$

which is Gauss constraint, or eq. (3.23). By incorporating this into the phase factor $\chi(\Sigma)$, we can write the functional:

$$\Psi_D(\Sigma) = \exp\left(i\frac{(eq)^2}{4\pi}\Omega_e(\Sigma)\right)\Phi_D(\partial\Sigma) \tag{3.29}$$

and we know have found two equivalent forms of wave functionals, eq. (3.25) and eq. (3.29) both satisfy the constraint (3.23). This result is expected as the dyon has both magnetic and electric charge.

Our constraint, equation (3.21), follows the same structure as the topological constraints examined in [15, 14]. We can utilize the fact that the "dice" derivative (i.e., closed surface derivative) can be rewritten as

$$\epsilon^{ijk}\partial_k\delta_{jk}(\vec{x}) = \frac{1}{3}\epsilon^{ijk}(\partial_k\delta_{ij}(\vec{x}) + \partial_j\delta_{ki}(\vec{x}) + \partial_i\delta_{jk}(\vec{x})) = \frac{1}{3}\epsilon^{ijk}\triangle_{ijk} \equiv \frac{1}{3}\triangle(\vec{x}).$$

Thus, we can rewrite the constraint in the following form:

$$\left(\frac{1}{3}\frac{i}{eg}\triangle(\vec{x}) + eg\delta^{(3)}(\vec{x}-\hat{\vec{r}})\right)|\Psi\rangle_{phy} = 0 \tag{3.30}$$

Applying this to wave functionals of the form $\Psi_D(\Sigma,\vec{r}) \equiv \exp(\chi(\Sigma,\vec{r}))\Phi_D(\Sigma,\vec{r})$, we must then satisfy:

$$\triangle(\vec{x})\chi(\Sigma,\vec{r}) = -3egi\;\rho_D(\vec{x}) \tag{3.31}$$

where $\rho_D(\vec{x})$ represents the charge density of the dyon (corresponding to the second term in the previous constraint). The function $\chi(\Sigma,\vec{r})$ can be expressed as follows:

$$\chi(\Sigma,\vec{r}) = -iK\int d^3x\int d^3x'\;\rho(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\epsilon_{ijk}\,\partial_i^{\vec{x}}T^{jk}(\vec{x},\Sigma) \tag{3.32}$$

which arises from our chosen surface representation and the tensorial nature of the constraint under consideration. Expanding this expression, we obtain:

$$\begin{aligned}
\chi(\Sigma,\vec{r}) &= -iK\int d^3x\int d^3x'\;\rho(\vec{x'})\frac{1}{|\vec{x}-\vec{x}'|}\epsilon_{ijk}\,\partial_i^{\vec{x}}\;T^{jk}(\vec{x},\Sigma)\\
&= -iK\int d^3x\int d^3x'\frac{1}{|\vec{x}-\vec{x}'|}(eg\delta^{(3)}(\vec{x}-\vec{r}))\epsilon_{ijk}\partial_i^{\vec{x}}\int d\Sigma_y^{jk}\delta^{(3)}(\vec{x}-\vec{y})\\
&= -iegK\int d^3x\int d\Sigma_y^{jk}\frac{1}{|\vec{x}-\vec{r}|}\epsilon_{ijk}\;\partial_i^{\vec{x}}\delta^{(3)}(\vec{x}-\vec{y})\\
&= +iegK\int d^3x\int d\Sigma_y^{jk}\partial_i^{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{r}|}\right)\epsilon_{ijk}\delta^{(3)}(\vec{x}-\vec{y})\\
&= +iegK\int d\Sigma_y^{jk}\partial_i^{y}\left(\frac{1}{|\vec{y}-\vec{r}|}\right)\epsilon_{ijk}\\
&= -iegK\int d\Sigma_y^{jk}\frac{(y-r)^i}{|\vec{y}-\vec{r}|^3}\epsilon_{ijk}\\
&= -2iegK\int dS_i\frac{(y-r)^i}{|\vec{y}-\vec{r}|^3}
\end{aligned} \tag{3.33}$$

Here, we note that the final term contains the solid angle subtended by the surface with respect to the dyon's position. The constant K remains to be determined.

Substituting $\chi(\Sigma,\vec{r})$ into the constraint equation (4.98), we obtain:

$$\triangle(\vec{y})\chi(\Sigma,\vec{r}) = \epsilon^{ijk}\triangle_{ijk}(\vec{y})\chi(\Sigma,\vec{r}) = 3\epsilon^{ijk}\partial_k\delta_{ij}(\vec{y})\chi(\Sigma,\vec{r}), \tag{3.34}$$

which expands as follows:

$$\begin{aligned}
&= -3iK\epsilon^{ijk}\partial_k^{\vec{y}}\int d^3x\int d^3x'\rho(\vec{x'})\frac{1}{|\vec{x}-\vec{x'}|}\epsilon_{lmn}\partial_l^{\vec{x}}\left(\delta_{ij}(\vec{y})T^{mn}(\vec{x},\Sigma)\right)\\
&= -i3K\partial_k^{\vec{y}}\int d^3x\int d^3x'\rho(\vec{x'})\frac{1}{|\vec{x}-\vec{x'}|}\epsilon_{lmn}\epsilon^{ijk}\partial_l^{\vec{x}}\frac{1}{2}\left((\delta_i^m\delta_j^n-\delta_j^m\delta_i^n)\delta^{(3)}(\vec{x}-\vec{y})\right)\\
&= -i6K\int d^3x\int d^3x'\rho(\vec{x'})\frac{1}{|\vec{x}-\vec{x'}|}\partial_k^{\vec{x}}\partial_k^{\vec{y}}\delta^{(3)}(\vec{x}-\vec{y})\\
&= +i6K\int d^3x\int d^3x'\rho(\vec{x'})\nabla_{\vec{x}}^2\left(\frac{1}{|\vec{x}-\vec{x'}|}\right)\delta^{(3)}(\vec{x}-\vec{y})
\end{aligned} \tag{3.35}$$

where in the last line we integrated by parts. Notice now that we can use the green function of the Laplacian, which has the property:

$$\nabla_{\vec{x}}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta^{(3)}(\vec{x}-\vec{x}') \tag{3.36}$$

Then we can write

$$\triangle(\vec{y})\chi(\Sigma,\vec{r}) = 24\pi iK\int d^3x\ \rho(\vec{x})\ \delta^{(3)}(\vec{x}-\vec{y}) = 24\pi iK\rho(\vec{y}) \tag{3.37}$$

Comparing with equation (3.31) we obtain that

$$K = -\frac{eg}{8\pi} \tag{3.38}$$

therefore writing the expression for the phase of the wave functional

$$\chi(\Sigma, \vec{r}) = i\frac{(eg)^2}{4\pi} \int d\Sigma_y^{jk} \frac{(y-r)^i}{|\vec{y}-\vec{r}|^3} \epsilon_{ijk} = i\frac{(eg)^2}{4\pi}\Omega(\Sigma). \tag{3.39}$$

This method is well constructed and can be generalized for higher dimension theories, as this approach does not require previous information about the nature of the system.

Two distinct geometric representations based on cycles can be established. The first is the standard or "direct" representation, where the cycles correspond to electric field lines and the magnetic field emerges as a derived quantity. The second is the "dual" representation, where the roles of electric and magnetic fields are reversed. In both approaches, the sources—whether electric or magnetic—can effectively be absorbed into a topological phase factor. This allows the charges to be "hidden" from the Hamiltonian and the dynamics, leading instead to the appearance of multivalued wave functionals [6, 9, 7].

Thus, the physical sector of the theory, which satisfies the generalized Gauss constraint, is described by multivalued cycle-dependent wave functionals. These functionals include a topological phase factor that captures the winding around the particle carrying the corresponding charge (electric or magnetic), regardless of which representation is chosen. In essence, the presence of electric or magnetic sources manifests through unusual boundary conditions that the wave functionals must satisfy.

It can therefore be said that, in the presence of a monopole, geometrically realized wave functionals naturally become multivalued.

Chapter 4

Conclusions

Conclusion

The initial chapters of this thesis provided a structured overview of key theoretical concepts, including field theory, canonical quantization, magnetic monopoles, and geometric representations. This background was essential for developing the central contribution of this work: a geometric formulation capable of describing *dynamic dyons*, particles that carry both electric and magnetic charge in motion.

Building on the established formalism for electric and magnetic monopoles, we extended the geometric representation of Maxwell's field to incorporate dyonic sources. This formulation allows for wave functionals that reflect topological phases induced by both types of charges, resulting in a generalized structure that goes beyond previous treatments limited to static or single-type sources. This work takes a step further by proposing the first formulation of dynamic dyons in the cycle representation framework — an advancement that has not yet appeared in the literature. The geometric quantization procedure employed here was grounded in the pioneering work of Leal, Fuenmayor, and Contreras [12, 14, 9, 6], whose foundational ideas were crucial for this development.

Our motivation to pursue this direction stems from the theoretical significance of dyons in high-energy physics. As Dirac famously showed, the existence of magnetic monopoles could explain the quantization of electric charge [1, 2], and theories admitting dyons offer promising avenues for unification and topological insights. Furthermore, recent developments in gravitational physics — particularly the construction of Black Bounce (BB) solutions to Einstein's field equations — have begun to incorporate not only magnetic but also dyonic sources [3], highlighting the increasing physical relevance of such configurations.

In summary, this thesis contributes a new step in the geometric quantization of dyonic systems. By framing dynamic dyons within a topologically rich representation, it opens the path for future work in areas ranging from gauge theory and gravitational models. A full formulation and publication of these results will follow as a natural continuation of this research.

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