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Gödel's theorem

.

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RESUMEN

En este trabajo se realizará una exposición del primer Teorema de Gödel. En esta exposición se detallará los preliminares necesarios para entender el teorema y se dará una prueba accesible y teórica más que técnica de cómo funciona la argumentación de Gödel del primer teorema. Aparte se verá un poco de la historia y contexto del Teorema.

ABSTRACT

In this work we are going to present Goidels First Theorem In this presentation we are going to present the necessary preliminaries for understanding Goidels first theorem and we are going to present a theoretical and accessible proof of how the theorem works, more than a technical aspect of the proof. Beside this we are going to see the history and the context of the problem

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INTRODUCTION

Context

To understand Gödel's theorems, we need to make clear or understand first a little bit of philosophy of mathematics and the historical concept of the problem. Gödel's theorems are in a sense a response to the Principia Mathematica of Bertrand Russel. Most importantly it is a response to the idea of trying to axiomatize all of mathematics by David Hilbert. An axiom is a statement that you accept as true without needing a proof. Besides axioms you have rules of inference. These are rules that tell you where you can go. For example a rule of inference may say that if you have the key A you have to open the door A. So if you have the key A then the door A is opened by you. Rules of inference in a sense are the rules that dictate where you can go from point A. The initial points are obviously the axioms. The combination of inference rules starting from axioms will be the theorems of the system, since you can prove that you can get to the theorem from the axioms if you follow the rules of inference. This example will be really helpful to understand this

Axioms

P,O,I

Rules of inference

1. If you have P then you can have PI
2. If you have X, X being any string of this symbols then you can have XOX
3. If you have IO you can have P

So for example let us start with the initial axiom P.

Since we have P by rule 1 we can have PI

By rule 2 we can have PIOPI

By rule 3 we can have PPPI

Another example

Let us start with axiom I

By inference rule 2 we can have IOI

By inference rule 3 we can have PI

In this example you can clearly see that our axioms are the three initial letters and the strings that we can derive through our inference rules are the theorems of the system. This is a basic example of what a formal system looks like.

Through history mathematics has work in this sense. For example, Euclidian geometry behaves in this way. There are 5 axioms in Euclidian geometry. From these five axioms you can derive theorems of geometry. For example, the Pythagorean Theorem is a combination of these axioms through rules of inference. Sadly, not in all systems the rules of inference have been explicit like in our little example system. Sometimes the rules of inference are human intuition and human logical sense. Mathematics hasn't always been formalized so that the rules of inference and axioms are explicit in the system.

During the 20th century one of the goals of mathematicians was to formalize mathematics, specially the mathematics that is concerned with properties of natural numbers. The objective of this was to have a finite number of axioms and explicit rules of inference, and once you define these ones, to prove that you can derive all theorems of the system. A theorem will be a true statement of the system. So, if we can derive all theorems we can derive all true statements of the system That way when a mathematical problem appears and someone wants to prove it, he can be sure that through a particular axiomatization and rules the theorem could be proven. Analogous to our example, theorems will be strings and we can get to any "true string" through our rules and axioms.

Since mathematics is expressed through symbols, we can easily assign a finite number of symbols to characterize the mathematical system. Sort of like a language with letters and punctuation symbols we can create infinitely many expressions that have meaning, we can create a mathematical alphabet for our system to create the mathematical expressions. That way a formal system will look alike our example but more complex. During this paper we won't explicitly do this but we will theoretically use this idea to prove Gödel's theorem.

An important aspect of Gödel's theorem is in the computational sense. A computer has programs that are algorithms that go from point A to point B through rules of language, analogous to our rules of inference. In that case we can find algorithms for all theorems and we can in a sense computerize mathematics.

Sadly, this project crumbled down when Gödel proved that in a formal system there are expressions or strings that are true but that can't be proven. This means that there are expressions that are true but can't be derive through rules of inference from the axioms.

We are going to try to understand his main idea in a more didactic way before going to the actual proof.

The main idea behind Gödel theorem is when the system is self-referential. For example, if I say "X always lies" and I replace X with any other person other than myself, for example X=Jorge. Then that sentence can either be true or false. But if I replaced X with the subject "I". Then we have the sentence "I always lie". Which if it is true that I always lie then I will be saying a truth which contradicts the statement. And if it was a lie than I will always tell the truth but right now I am telling a lie. So, we have a paradox. So, when a system becomes self-referential then we have these types of sentences that are neither true nor false.

Of course, in mathematics formulas often talk about numbers similar to when in language one talks about other object or people. But Gödel found a way of making a mathematical statement refer to itself and that way construct the paradox of language in a mathematical sense.

History

During the 20th century one of Hilbert's goal was to axiomatize all of mathematics. Gödel heard about Hilbert's project through his thesis Professor who encouraged him to dedicate himself to this problem. In 1930 Gödel solved two important results. The first one being that there are true statements in a formal system that are not provable. This statement later became that a system is incomplete. This theorem is known as the first incomplete theorem. The second problem he solved was that a formal system cannot determine its own consistency. This is Goidels second theorem and one of Hilbert's 23 problems.

Gödel's first exposition of his theorem was given in Konigsberg. His lecture was about the incompleteness of the Principia Mathematica a famous book by mathematician Bertrand Russell. His lecture didn't produce any controversy in the public. His results were not clearly understood by the people. There was one famous mathematician Jon Von Neuman who understood the gravity of the results and later popularized them. It is said that Hilbert was deeply frustrated with Gödel's results. His results are controversial in the mathematical world. People who are experts on them say we still don't grasp the gravity of his results

The following phrase was said in the lecture and it summarizes the lecture itself

"Assuming the consistency of classical mathematical systems one can give examples of prepositions that, while contentually true, are unprovable in the formal system of classical mathematics."

Gödel later proved that the continuum hypothesis is true but not provable in the Zemelo-Fraenkel(ZF) axiomatic system. The ZF system was an axiomatic system to represent all of set theory. It didn't work for Gödel. He later created his own axiomatic system for set theory.

Gödel scape WW2 by receiving a US visa. He taught in Princeton. He died in 1978. He suffered mentally and physically his last ten years. He suffered depression and paranoia to the point he wouldn’t eat because he was afraid people will poison his food. He died of malnutrition.

DEVELOPMENT

Abstract proof

Since Gödel's theorems deals with formal systems we have to define what are the characteristics of a formal system. The way we are going to define a formal system for this proof is to assimilate it to a language L. Our language has a way of representing natural numbers and infinitely many variables with finite characters. For example, let us suppose C stands for 0 then C' stands for 1 and so on. Also, with variables suppose we have two variables in a sentence then V with be the first variable and V' the second and so on. So, we are going to have a language composed of finite letters or characters. Since we have finite characters the number of combinations we can make with these characters is countable. Since for expressions that only have one letter we have the number of expressions equal to the number of characters. For expressions of two letters we also have finite expressions, and so on. So, we have countable expressions for our system L. That will be our first set. The set E of expressions, and here is the structure of a formal system.

Structuring of a Formal Language

1. Set E of Expressions
2. The second set is a subset of E called S that are the sentences of E. (In a particular system a sentence is a combination of characters that don't violate the rules or intern logic of the system. Is like in a language it is a combination of symbols that is *syntactically correct. For example, in a language you can say the sky is green without being true. But you can't say green sky is the because it has no logic. Our formal system will also have a logic that explicitly sates the rules of arrangement. Every sentence has a true value to it)*

3. The third subset is a subset of S called the provable sentences P (Sentences that we can derive through the rules of the system from the axioms)
4. The fourth is the subset of S called R that are the refutable sentences (Sentences that their negation is a provable sentence)
5. A subset of expressions called predicates H of L (A predicate is an expression that has a variable that can be replaced by a natural number, when it is replaced we have a sentence)
6. The last subset will be the subset T that will be a subset of S called the true sentences of L. The definition of truth for a sentence will depend on the rules of logic of the particular system. For this abstract scheme we only point out the existence of sentences that are true. Note that Ture is not the same as provable. Provable statement is that through a deductive system with the axioms and the rules of inference we can arrive to the statement. A true sentence for example is one that once we define what means for a system to be truth it then fulfills the definition. But this definition doesn't come from the system itself it comes from another system, a meta system that defines what it means for a sentence to be true in the system we are working. This is the relationship object language (language we are working) and meta language (language that describes the object language). This is Tarski's theory of truth in a nutshell. I am not going to prove or go into further detail. But even Gödel proved that arithmetic truth can't be defined from arithmetic itself. Just clarify this so the reader can understand that the notion of truth is different from the notion of provable.

   Another way of thinking about it, is to think of true sentences as the truths of mathematics that live in the platonic realm of the world of truths of mathematics. We humans have constructed formal systems to grasp these platonic truths, but to have reached finitely many truths with formal systems doesn't mean that we can reach to all of these platonic truths in the platonic truth world.

What mathematicians before Gödel wanted to proof is the all True sentences are provable and all provable sentences are true. Mainly mathematicians wanted to prove we can arrive to the platonic truth world. So, they have to be different sets as an initial assumption and then we need to proof they are the same. But Gödel found that this was not the case.

7. Lastly, we have a function α that associates each Expression with a natural number creating the expression E(n). If the expression is a predicate we have a sentence. Also, we have sentences without variables so if that is the case E(n)=E for all n. But if a variable is in there then we have a different expression for each E(n) and this is specially the case for predicates. Note that we have countable many predicates and countable many integers. From set theory we know that NxN is also countable so we still have countable expressions of the form E(n)

If you note our system besides having these 7 subsets it also needs to have some other properties. For example, it needs to be referring to natural numbers. It needs to have the negation operator in the language. Also, for the notion of truth it needs the equal and less than or equal operator. These will be clearer as we go through but it is just a clarification for right now.

This is the structure of what a formal system looks like form an abstract way.

We are going to observe some definitions that will help us in the process. Also, every set that refer right now is going to be talking about a set that belongs to the natural numbers

Definitions

Expressability

Set A is said to be expressed by a predicate H if H(n) is true for all n in A and false for all n not in A. So we have

$H(n) \in T \leftrightarrow n \in A$

Expressablility: a set A is expressible in the language L if A is expressed by some predicate of L.

Correct system

A system is said to be correct if every provable sentences of L are true and all refutable sentences are false. So, P is a subset of T and T and R are disjoint. We can only arrive to true sentences. The sentences we can't arrive are refutable sentences and we can proof they are refutable by arriving to their negation Note that a true sentence and provable sentence are not strictly the same. Later this distinction will become clearer

Gödel Numbering

Since we have countable many expressions we can have a 1 to 1 function from the expressions to the natural numbers. This number will be the Gödel number of the Expression. Note that if we have a predicate H that predicate has a Gödel number and the Gödel number of that predicate is different from the Gödel number of the specific sentence H(n). One thing is a predicate with a variable that has its own Gödel number and another different is the sentence H(n). They are both different expressions. The following chart will explain this more graphically.

Supposed we have all predicates. Since there are countable expressions there are countable predicates. Since each predicate has a Gödel number we can organize them in numerical values from least to greater according to their Gödel number. The Gödel number of predicate H_{gi} is gi. Then we make a table.

	g1	g2	g3	g4	g5	g6	g7	g8	g9
$H_{g1}(n)$	$H_{g1}(g1)$	$H_{g1}(g2)$	$H_{g1}(g3)$	$H_{g1}(g4)$	$H_{g1}(g5)$	$H_{g1}(g6)$	$H_{g1}(g7)$	$H_{g1}(g8)$	$H_{g1}(g9)$
$H_{g2}(n)$	$H_{g2}(g1)$	$H_{g2}(g2)$	$H_{g2}(g3)$	$H_{g2}(g4)$	$H_{g2}(g5)$	$H_{g2}(g6)$	$H_{g2}(g7)$	$H_{g2}(8g)$	$H_{g2}(g9)$
$H_{g3}(n)$	$H_{g3}(g1)$	$H_{g3}(g2)$	$H_{g3}(g3)$	$H_{g3}(g4)$	$H_{g3}(g5)$	$H_{g3}(g6)$	$H_{g3}(g7)$	$H_{g3}(g8)$	$H_{g3}(g9)$
$H_{g4}(n)$	$H_{g4}(g1)$	$H_{g4}(g2)$	$H_{g4}(g3)$	$H_{g4}(g4)$	$H_{g4}(g5)$	$H_{g4}(g6)$	$H_{g4}(g7)$	$H_{g4}(g8)$	$H_{g4}(g9)$
$H_{g5}(n)$	$H_{g5}(g1)$	$H_{g5}(g2)$	$H_{g5}(g3)$	$H_{g5}(g4)$	$H_{g5}(g5)$	$H_{g5}(g6)$	$H_{g5}(g7)$	$H_{g5}(g8)$	$H_{g5}(g9)$

Obviously, this table goes to infinity down and to the right. Where gi represents the Gödel number of the predicate and then we evaluate each predicate. But the Gödel number of H_{gi} as a predicate is different form $H_{gi}(n)$ for all n.

Note that in statements without variables (sentences) the Gödel number is always the same, and this table can't be made since this table is for predicates

What we are interested in this table is what happens to expressions of the form $H_{gi}(gi)$. This is called the diagonalization of $H_{gi.}$ This sentence will be true if and only if the sentence expresses the Gödel number of the predicate itself.

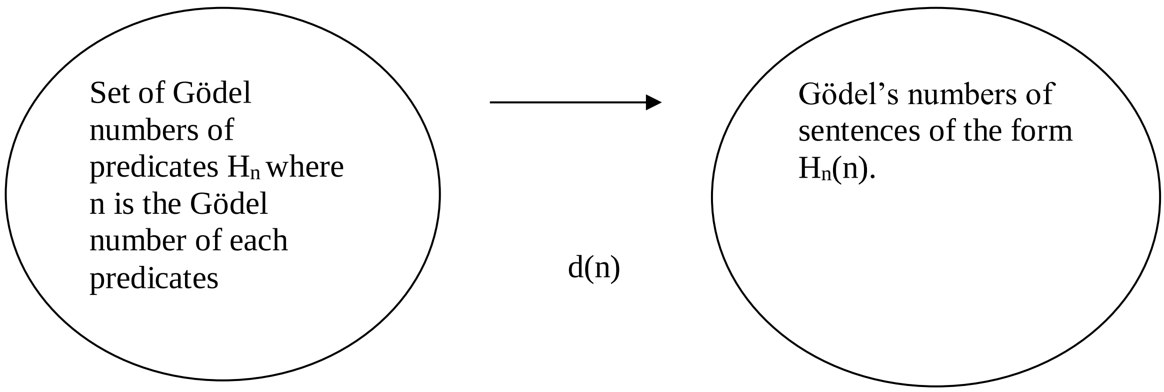
Note that doing this is like talking about ourselves. We associate each predicate with a number and then that predicate talks about the number is associated. Is like saying something about ourselves i.e "I always lie"

Diagonal function

Let us define diagonalization more formally. The diagonalization of an expression H_{gi} is the expression $H_{gi}(gi)$. If the expression is not a predicate then the diagonalization is the same expression. But we are mainly interested in predicates

We call the function d(n) the diagonal function of the system. Which takes any natural number $n \in N$ and makes it the Gödel number of $H_n(n)$. It is a function from the naturals to the naturals. That in other words is sending the Gödel number of the predicate H_n to the Gödel number of the diagonalization of $H_n(n)$. It actually takes the Gödel number of any expression even if it is not a predicate. But since the diagonalization of a non-predicate expression is itself then d() of a non-predicate Gödel number is the same Gödel number. But we are only interested in predicates. We can forget about non-predicates.

The following two graphs will explain the relationship easier.



As you can see it is sending the Gödel number of each predicate to the diagonal. That is why it is the diagonal function.

	g1	g2	g3	g4	g5	g6	g7	g8	g9
$H_{g1}(n)$	$H_{g1}(g1)$	$H_{g1}(g2)$	$H_{g1}(g3)$	$H_{g1}(g4)$	$H_{g1}(g5)$	$H_{g1}(g6)$	$H_{g1}(g7)$	$H_{g1}(g8)$	$H_{g1}(g9)$
$H_{g2}(n)$	$H_{g2}(g1)$	$H_{g2}(g2)$	$H_{g2}(g3)$	$H_{g2}(g4)$	$H_{g2}(g5)$	$H_{g2}(g6)$	$H_{g2}(g7)$	$H_{g2}(8g)$	$H_{g2}(g9)$
$H_{g3}(n)$	$H_{g3}(g1)$	$H_{g3}(g2)$	$H_{g3}(g3)$	$H_{g3}(g4)$	$H_{g3}(g5)$	$H_{g3}(g6)$	$H_{g3}(g7)$	$H_{g3}(g8)$	$H_{g3}(g9)$
$H_{g4}(n)$	$H_{g4}(g1)$	$H_{g4}(g2)$	$H_{g4}(g3)$	$H_{g4}(g4)$	$H_{g4}(g5)$	$H_{g4}(g6)$	$H_{g4}(g7)$	$H_{g4}(g8)$	$H_{g4}(g9)$
$H_{g5}(n)$	$H_{g5}(g1)$	$H_{g5}(g2)$	$H_{g5}(g3)$	$H_{g5}(g4)$	$H_{g5}(g5)$	$H_{g5}(g6)$	$H_{g5}(g7)$	$H_{g5}(g8)$	$H_{g5}(g9)$

Now we are going to define two sets in the natural numbers A and A* such that for every natural number n, $d(n) \in A$ and $n \in A^*$. In other words, $d^{-1}(A)$ is A*. Note that we are defining A* in terms of A and not the other way around. But since the diagonal function is one to one then the following relationship still hold

$n \in A * \leftrightarrow d(n) \in A$

We can also say that this relationship is between the Gödel numbers in the diagonal and the Gödel numbers of the predicate.

Now we are ready to use all of these definitions and start proving Gödel's First Theorem. Besides all of these definitions we need a little bit of basic set theory. Another important aspect of this first sketch of the proof is that we are going to make some assumptions. With the definitions we have and the assumptions we are going to make it will be enough to proof the theorem and start seeing the general idea behind Gödel's thinking. After this proof we are going to proof the assumptions in a more technical way. After proving the assumptions that we are going to use right now we will understand that these assumptions are actually necessary condition for the type of formal systems to which Gödel's first theorem applies.

Assumptions

A_1 For any set A expressible in L, the set A^* is expressible in L

A_2 For any set A expressible in L, the set $\bar{A}$ is expressible in L. By $\bar{A}$ we mean the complement of A

A_3 The set P is expressible in L. Where P is the set of Gödel numbers of all provable sentences.

We are going to see later that Assumptions one and two are actually hinted at the phrasing of the formal expression of Gödel's First theorem

But first we need some definitions to get the idea of the statement of Goidels first theorem

Definitions

Consistency

A system L is said to be consistent if no sentence is provable or derivable and refutable or not derivable at the same time. It is called inconsistent if a sentence is provable and refutable at the same time.

Note that a correct system is consistent since T and R are disjoint and P is a subset of T. So, correctness implies consistency

Completeness

To define completeness, we first need to define decidability. A sentence X is decidable if it is provable or refutable. A system is complete if all sentences in the system are decidable. Note provable and refutable is not the same as true and false

Now we can state the theorem

"Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F." (Raatikainen 2015)

Assumption two is implicit in **consistent formal system**. And Assumption 1 and 3 are implicit in **certain amount of elementary arithmetic**. But we will see this when we proof assumptions one two and three.

Proof itself

The proof we are formulating is not the original one made by Gödel. It is a later proof with concepts of Tarsky. Having said that let us continue with our proof.

Since P is expressible in our correct system L then by our two assumptions.$\tilde{P}^*$ is expressible in L. By assumption 2 P complement is expressible in L. By Assumption 1 applied to P complement $\tilde{P}^*$ is expressible in A. So, we have that $\tilde{P}^*$ is expressible in our correct system L.

Let H a predicate that expresses $\tilde{P}^*$ in our correct system L. Let h be the Gödel number of this predicate H. Since H expresses $\tilde{P}^*$ then H(n) $is\ true\ if\ and\ only\ if\ n \in \tilde{P}^*$. Let us see what happens to the diagonalization of our predicate H, H(h). H(h) $is\ true\ if\ and\ only\ if\ h \in \tilde{P}^{*}$.

But

$h \in \tilde{P} * \leftrightarrow \mathrm{d(h)} \in \tilde{P} \leftrightarrow \mathrm{d(h)} \notin \mathrm{P}$

This means that H(h) is true if and only if the diagonalization of d(h) is not in P. This will mean that H(h) is true but since its Gödel number is not in P then H(h) is not a provable sentence. That will mean that there are true sentences that are not probable or derivable.

Just to avoid this let us then suppose H(h) is false. Then that will mean that d(h)∈P. Then this will mean that H(h) is false and derivable. This is against the definition of a correct system. So we can't have this either

So, what we can have at the end is that H(h) is true but not derivable. Which means that there exist true sentences that can't be proven in our system.

We have proven Gödel's first theorem that there are true sentences that can't be proven so our correct system is incomplete because these statements are true and can't be proven and T and R are disjoint so these statements are undecidable.

We still need to proof our 3 Assumptions and we need to clarify the notion of truth a little bit. This will be done later in a technical way using examples of how to do it for a particular formal system.

To continue we are going to proof that there are true sentences that are not provable in a different way. For this we are going to introduce the concept of Gödel Sentence

Gödel Sentence

A sentence E_{n} is a Gödel sentence for the set A if either, E_n is a true sentence and n (The Gödel number of the sentence) is in A. Or if it is false then Gödel number is not in A. With this definition we can now prove a Lemma

Diagonal Lemma: For any set A, if A* is expressible in L the there is a Gödel sentence for A. Since L satisfies condition or Assumption 1, A_1, then there is a Gödel sentence for every expressible set A.

Proof Of Lemma

Let H a predicate that expresses A^* in our correct system L. Let h be the Gödel number of this predicate H so d(h) is the Gödel number of H(h). Since H expresses A^* then

H(h) $is\ true\ if\ and\ only\ if\ h \in A^*$. Then

$h \in A * \leftrightarrow \mathrm{d(h)} \in A$

So, H(h) is true if and only if its Gödel number belongs to A. Or it is false if and only if its Gödel number doesn't belong to A. So, H(h) is a Gödel sentence by definition.

Now let us proof again that there exists true sentences that are not provable with the notion of a Gödel Sentence.

Second Proof of Gödel's Theorem

By our three assumptions $\tilde{P}^*$ is expressible in L. This means that there is a Gödel sentence for $\tilde{P}$ by the Diagonal Lemma. This will mean that the Gödel sentence for $\tilde{P}$ is true if and only if its Gödel number is not in P. So, the Gödel sentence number will not be in the set of provable sentences of L so it is true and not probable. Or it is false and its Gödel number belongs to P and then we have an incorrect system. We have proven again that there are true sentences that can't be proven.

The notion of Gödel sentences will also help us talk a little bit about the set T of true sentences of L.

True Sentences properties

First, we define the set of natural number T to be the set of Gödel numbers of true sentences.

Now we have the following three theorems.

1. The set $\tilde{T}*$ is not expressible in L
2. Since $\tilde{T}*$ is not expressible in L. The set $\tilde{T}$ can't be expressible in L because of $A_{1.}$ So, if the condition holds it will violate theorem 1
3. Since $\tilde{T}$ is not expressible in L then T can't be expressible in L by $A_{2.}$ So, if the condition holds it will violate theorem 2

So, we only need to proof the first theorem and the other two will be proven thanks to the conditions that we assume.

Proof

If $\tilde{T}*$ is expressible in L, then there is a Gödel sentence for $\tilde{T}$. This cannot be the case since that will mean that if the Gödel sentence is false its Gödel number belong to T and this is absurd. Also, if it is true then its Gödel number doesn't belong to the set T, which is also absurd.

So, we have finally that $\tilde{T}*$,$\tilde{T}$ and T are not expressible in L

Note that P is expressible in the system by Assumption 3. So, we can see once again that T and P are not the same since T is not expressible and P is.

As you can see our proof is for correct systems. But since correctness implies consistency we can be sure our system is consistent.

Summary

We have used the technique of Gödel numbering to keep track of sentences. The type of sentences that we are interested in are diagonal sentences. These sentences have an interesting property that in a sense they are talking about themselves. Also, when the system follows our three assumptions that we made the first two being conditions of a formal system, then we have found that the diagonal sentences that we have formed are undecidable. We have found undecidable sentences that are perfectly well formed just by assuming that the set P is expressible and then following A_1 and A_2. We still need to proof these three assumptions and that is what we are going to do right now in a more technical and particular way. But first we need some preliminaries.

Preliminaries.

The first preliminary is to understand that in a more specific language we don't have expressions we have formulas. Our symbols represent mathematical relationships such as = or less than or equal etc. A formula is a statement that is syntactically correct in the system and that it expresses one of the two relationships ≤ or =. To create a specific system for this paper is not necessary since we are only interested in the proof in general but we need to understand how a formal mathematical language will behave to get the idea.

To represent numbers and variables we need also a symbolic system. Even though to represent numbers we already have finite symbols we can make it with fewer symbols. For example, we can have 0 as itself and 0´ as one and 0´´ as two and that way we have all the natural numbers with only two symbols.

Now we are ready to introduce the concept of formulas and sentences in a more particular system. Note that a formula will be the same as a predicate in the abstract proof that we did and a sentence will be a formula that is already replace with a natural number or natural numbers that the formula needs. Let us imagine that our formulas have the structure of

$$2x + 3y = 5$$

So the formula will have two variables F(x,y). To represent the first variable we can use a character like v and to represent the other ones we can have v´,v´´,v´´´ etc. Like in our more abstract example sentences have syntactical restrictions also formulas have them. Mainly a formula is an expression that show an equal relationship or a less than or equal relationship between numbers.

Lastly an important condition of the more particular system is to have characters for the elementary arithmetic operations +, * and exponentiation. And logical operator not∼

Notion of truth in the particular system

As we already said the notion of truth depends on the system. But here we are going to make a little acclamation that will help us differentiate between provable and true. In a correct system every provable statement is true as a definition. But we haven't quite defined what it takes for a sentence to be true we only mention that True and provable are not the same. It is easier to explain this in a more particular mathematical system. Truth statements are statements that once defined the "=" and "≤" relationship, these relationships are true. This is why the notion of truth is related with sets of numbers. Because these relationships in a predicate are true depending on the set of numbers that make them true. So even if a predicate describes a relationship that is true for a set of numbers, that doesn't mean that from the axioms we can prove it. This was just a clarification to understand better the notion of truth that is easier to understand in a particular example.

Note that the notion of truth doesn't come from arithmetic itself but by us explaining the relationship of numbers. We are doing a meta language to our formal system.

Arithmetic

Let us have a formula F(v) that is true for the elements of set A. Then A is arithmetic. It is the same as expressability in our more abstract example. So now we have this more precise definition

$F(n) \in T \leftrightarrow n \in A$

Then the set A is called arithmetic

Since in this particular system we have formulas with more than one variable then we say the vector made of natural numbers (N1,N2,N3..Nn)) is expressed in a formula then the vector is arithmetic. So, a vector is arithmetic if the relationship between numbers in the vector and the formula make the formula true.

As you can see it is very similar to our definition of expressible.

Lastly, we are going to introduce a new concept called an arithmetic function

A function (form R^N to N) f(x1,x2,x3,4..xn) is arithmetic if the relationship f(x1,x2,x3,4..xn)=y is arithmetic. For this relationship to be arithmetic it means that there is a Formula F(v1,…vn, vn+1) that when replaced with x1..xn1,y the formula is true. So, the formula is true if and only if the relationship of the function with its arriving value holds.

In a sense what this function is telling us is that there is a way of taking a formula in our language and transforming it to an arithmetic relationship of a function with an arriving value. Since functions are arithmetic they take values and use arithmetic operators, in a sense we are making the formula arithmetic and we are also making arithmetic relationships into formulas in our language.

Gödel numbering in our system

We are not going to do a whole system for creating Gödel numbers for this particular system. But we are going to list three properties that the Gödel numbering should have

1. The Gödel numbering is a process in which we assign a number to each character of our system and using arithmetic operators in the expression made of these characters we get the Gödel number of our expression from the associated number of our characters being transformed from arithmetic operators
2. For any n being a natural number in our system, there is an arithmetic function f(n)=z, where z is the Gödel number of n. So, the Gödel number of n is represented in the relationship strictly depending on n. Also, since we can create an arithmetic function we can create a sentence that is saying “there exists an n such that if we use the

process of Gödel numbering in n it will give us the Gödel number of n that it is z." in our mathematical language. This sentence that we have created is the one that expresses n and z

3. For any two expressions (expressions not formulas) X and Y, with Gödel numbers x and y, there exists an arithmetic function f(x,y)=z such that z is the Gödel number of XY
4. For any formula X with Gödel number x and for any natural number n with Gödel number y, there exists an arithmetic function such that r(x,y)= the Gödel number of X(n). This means there is a formula R(x,y,godel number of X(n)) that expresses x, y and the Gödel number of X(n)

Note that in properties 2, 3 and 4 we are not saying anything about the truthfulness of the Formulas or expressions itself. We are saying truthfulness about the formulas that construct their respective Gödel numbers with the arithmetic functions we created. The expressions may be true or false but the Formulas that talk about the Gödel numbers that we have created using arithmetic properties of the Gödel numbering system are true for our arithmetic function relationship.

Diagonalization

Since we already defined an arithmetic function r(x,y)=z where x is the Gödel number of the formula F_x and y is the Gödel number of the natural number y and z is the Gödel number of $F_x(y)$, then we just use as a diagonal function the function r(x,x)=d(x)=c where c is the Gödel number of diagonalization . So, we already have an arithmetic function for the diagonalization. This also means we have a formula expressing x and c.

Now we are ready to prove Assumption one of a formal system that we haven't proved

If A is arithmetic then A* is also arithmetic

Remember that $A^*=d^{-1}(A)$

So, we can say that A is arithmetic which means there is a Formula F(y) expressing y for y∈A. Also, there is a formula for the arithmetic function r(x)=y∈ A r being the diagonalization of x∈A*. So there are two formulas one expressing D(x,y) and the other one F(y). So we can construct in our language a formula saying “There exist an x∈A* such that D(x,y) and F(y)”. We make y constant with a value y that belongs to A and we have a formula that only involves x. This formula will be easy to construct and this x has to exist by the definition of our sets A and A* and it expresses A*. And this formula gives true for all values of A* when we change the constants for other y in A. So, A* is expressible.

We have proven Assumption one. Note that this assumption has to do in a system that contains arithmetic. That is why Assumption one is associated with systems with sufficient arithmetic. Because we need to create this correspondence between formulas and arithmetic functions.

Proof of Assumption 2

Assumption 2 is easy to prove. If the system is consistent and A is expressible then $\check{A}$ is expressible by the negation of the formula. So, if X expresses A not X expresses$\check{A}$.

So, all of our theorems from the abstract part have been proven for example for every arithmetic set A there is a Gödel sentence for A

Proof that True sentences are not arithmetic.

Since A is arithmetic so is A*. Let H(x) be a formula expressing A* and h the Gödel number of it. Then Since H expresses A^* then H(h) $is\ true\ if\ and\ only\ if\ h \in A^*$. Then

$h \in A * \leftrightarrow \mathrm{d(h)} \in A$

So, H(h) is a Gödel sentence for A

We can apply our assumptions that are already proven to the theorems we proved in an abstract way and now we can arrive to the conclusion that

The set T of Gödel numbers of true sentences is not arithmetic.

The only missing part to have completed the proof is to proof that the set P of Gödel numbers of provable sentences is arithmetic

Proving this is really technical and complicated that is why with a main idea of the process will be enough to understand the proof.

Proof of Assumption 3

To prove this assumption, we need to define the Gödel number of a sequence. Like in mathematics a proof is a sequence of statements and formulas that through a deductive way take you from point A to point B. We already know each sentence and formula composing the proof has a Gödel number. So, what we can do is create a Gödel number for the whole proof, a Gödel number for the whole sequence. Let's say that we have 5 strings that compose our proof. Each one with Gödel numbers A, B, C, D, E. The we can say that the Gödel number of this sequence will be a multiplication of the first prime numbers each one raise to the power of the Gödel number according to the order of the sentence. For this example, it will be

$K=2^{A}*3^{B}*5^{C}*7^{D}*11^{E.}$

Note I am doing the definition of the Gödel number of the sequence this way just to illustrate the idea. There are many different ways of doing them.

This will be the Gödel number of the sequence.

The last sentence is the one that we are mostly interested since it is a provable sentence with a Gödel number E.

We can see that there is an arithmetic relationship between K the Gödel number of the sequence and E the Gödel number of the provable sentence. So, we can define a function f(E)=K. Then we can create a formula in our system that states this arithmetic relationship of K and E. Finally, we have the formula F (V1, V2) which is true for V1=K, V2=E. We can make a new formula where V1=K and we have our new formula F(V2) which expresses E. So, every provable sentence is arithmetic since we can repeat this for all provable sentences. We have proven our three assumptions and with that we can be sure that our proof of Gödel's theorem is already completed.

Conclusions

Gödel's theorems are highly controversial in the mathematical world. What to me is interesting is that Gödel never doubt about the truthfulness of mathematics. Gödel was a Platonic mathematician, he believed higher mathematical truths exist in a higher dimension and we try to grasp them through reason and formal systems. But these truths exist independent of the system. His work has been taken form a postmodern perspective in which they try to argue that even mathematics can't determine what is truth and truth is only a perspective and not an absolute. He never got in that train. As Gödel I also believe mathematics as higher truths. But what it scares me is that we won't be able to fully understand these truths. We may have to accept our human condition and live in the cave and the shadows Plato talked about in his myth.

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