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Light Deflection around a Spherical Rotating Black Hole to fifth order. Lindstedt-Poincaré and Padé approximations.

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RESUMEN

En esta disertación se estudia a detalle el movimiento de partículas empleando la métrica de Kerr, es decir, donde la geometría del espacio tiempo está dada por la influencia de una fuente de gravitación que tiene un momentum angular. Específicamente, nos concentramos en buscar una aproximación al ángulo de desviación de la luz alrededor de agujeros negros rotantes por medio del método perturbativo de Lindstedt-Poincaré y la aproximación de Padé para funciones reales.

Palabras clave: Padé, Lindstedt-Poincaré, Kerr, agujero negro, desviación de la luz.

Abstract

In this dissertation the motion of particles is studied using the Kerr metric, where the geometry of space-time is given by the presence of a source of gravity which carries a net angular momentum. Specifically we work on developing a method to approximate the angle of deviation of light around rotating black holes. The Lindstedt-Poincaré perturbative expansion and the Padé approximation for real functions were employed to perform this task.

Key words: Padé, Lindstedt-Poincaré, Kerr, black hole, light deviation.

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Chapter I

Introduction

Albert Einstein published the General Theory of Relativity (GTR) in 1915, thereby generalizing Newton's gravitational theory to a relativistic scope [1]. This is a powerful and elegant formulation that describes a gravitational field and its equations. Among its most notable predictions, we have the gravitational redshift, apsidal precession, gravitational time dilation, deviation of light and gravitational lenses. Fundamentally, GTR states that there is an equivalence between the gravitational mass and inertial mass is also considered, which is called the principle of equivalence, it explains why a free falling body is not able to differentiate between acceleration and gravity. For example, someone falling in vacuum cannot know if it is at rest in a gravitational field or being accelerated by an external force.

the principle of covariance, which dictates that all observers, inertial or not, experience the laws of physics in the same way. This theory also generalizes the laws of physics to a curved space-time, which work for a flat space-time as well, this is the principle of minimum gravitational coupling. In 1885, Loránd Eötvös demonstrated this principle with great precision, using a torsion scale; initially he obtained results with an error of order 10^{-9}, future experiments have shown further increase in precision [2].

The General Theory of Relativity is described by fourteen equations, ten of which are Einstein's field equations, and the remaining four are the geodesics. These equations describe the curvature of space-time, its geometrical properties and the interaction between matter and energy. Einstein's field equations can be solved to find the metric of space-time for some distribution of matter and energy. They are written in tensor form as follows [3, 4]:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} + \lambda g_{\mu\nu}. \tag{1.1}$$

Here $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar: $R = g^{\mu\nu}R_{\mu\nu}$, $g_{\mu\nu}$ is the metric tensor; λ is the cosmological constant, a factor related to the expansion of the universe and the contribution of dark energy, and $T_{\mu\nu}$ is the energy-momentum tensor. Moreover, the form of equation (1.1) is given by the metric sign convention $(+---)$. Next, the equations of the geodesics are shown, they describe the motion of a particle that is immersed in a gravitational field:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\left(\frac{dx^\nu}{d\tau}\right)\left(\frac{dx^\rho}{d\tau}\right) = 0, \tag{1.2}$$

where Γ are the Christoffel symbols, and they are given by [3, 4]:

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\alpha}\left\{\partial_\sigma g_{\rho\alpha} + \partial_\rho g_{\sigma\alpha} - \partial_\alpha g_{\rho\sigma}\right\}. \tag{1.3}$$

In these last two equations, τ is the proper time (time measured by an observer following a world line in spacetime), and x^μ is the four-vector that describes the position. The simplest solution to Einstein's equations was given by Karl Schwarzschild in 1916 [5], which is able to describe a static and neutral black hole with spherical symmetry. The solution has the following form [3]:

$$(ds)^2 = c^2(d\tau)^2 = \gamma c^2(dt)^2 - \frac{1}{\gamma}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2, \tag{1.4}$$

where r, θ, ϕ are the coordinates of some point in space, t is the coordinate time (time measured by a distant observer) and $\gamma = 1 - \frac{2GM}{rc^2}$. The Schwarzschild solution is a powerful tool to understand the existence and formation of black holes. Nonetheless, it is common to find rotating distributions of matter in the universe, which is a fundamental characteristic that manifests from the subatomic to the astronomical scale. Therefore it is necessary to study a geometry that takes into account the rotation of mass in a curved space-time. The Kerr metric is an exact solution to Einstein's field equations, which allows the study of rotating black holes and the phenomena that can occur in this geometry of space-time. The expression for the arc length that gives the Kerr solution is as follows [6][3][7]:

$$\begin{aligned} ds^2 = &\left(1 - \frac{2\mu r}{\rho^2}\right)(c\mathrm{d}t)^2 + \frac{4\mu a r\sin^2\theta}{\rho^2}c(\,\mathrm{d}t\,\mathrm{d}\phi) \\ &- \frac{\rho^2}{\Delta}(\mathrm{d}r)^2 - \rho^2(\,\mathrm{d}\theta)^2 - \left(r^2 + a^2 + \frac{2\mu r a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta(\mathrm{d}\phi)^2, \end{aligned} \tag{1.5}$$

where $\mu = \frac{GM}{c^2}$, $a = \frac{S_z}{Mc}$ is the spin angular momentum per unit mass, $\Delta(r) = r^2 - 2\mu r + a^2$ and $\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$.

The deviation of light is one of the observational tools that serve to quantitatively study the geometry around a gravitational source, such as a black hole. The study of the gravitational effect on light rays is observed for instance in gravitational lenses [4], when a photon passes through a gravitational field, it acquires a deviation angle [8]. This phenomenon has been observed with the Sun's gravitational field, and several galaxies [9]. To study the deviation of light, it is possible to use approximation methods such as a series expansion to solve the equations of motion [10]. In 2020, Marín and Poveda published a similar study for a charged black hole [11]. In this work, the same method will be employed to study the deviation of light around a rotating black hole. Using the Lindstedt-Poincaré method to calculate the angle of deviation of light to fifth order. The results will be compared to those obtained from the Padé method of approximation by rational functions, which, for instance, has been used in cosmology to parametrize the luminosity distance, with very promising results [12] [13].

Chapter II

Motion around the Kerr geometry

In 1963 Roy Kerr found a solution to Einstein's equations, which describe a rotating gravitational source, devoid of charge. A massive object, described by Kerr's metric has two characteristic parameters, mass and spin angular momentum [3]. From equation (1.5), (Kerr's solution in Boyer-Lindquist coordinates), we can extract the metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2\mu r}{\rho^2} & 0 & 0 & \frac{2a\mu r\sin^2\theta}{\rho^2} \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{2a\mu r\sin^2\theta}{\rho^2} & 0 & 0 & -\frac{\sin^2\theta}{\rho^2}\Sigma^2 \end{pmatrix}, \tag{2.1}$$

where $\mu = \frac{GM}{c^2}$, $\Sigma^2 = \left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta$ and $a = \frac{S_z}{Mc}$, S_z being the spin angular momentum. Note that $g_{33} < 0$ and $g_{03} = g_{30} > 0$. In addition, the contravariant form of the metric tensor can be calculated by inverting $g_{\mu\nu}$. In the next section we will study the equations of motion in the equatorial plane, thus we are going to assume $\theta = \frac{\pi}{2}$ for the remainder of this work.

2.1 Equations of motion

The rotation of a very massive object, say a Kerr black hole, causes a very complex geometry of the spacetime around it. Any particle outside the equatorial plane would be forced to move between planes, this is because the Kerr spacetime is not spherically symmetric. For simplicity, we are going to consider motion in the equatorial plane, namely $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$. In this section, the equations of motion of test particles are derived using the Lagrangian formalism.

The Lagrangian is defined as:

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}, \tag{2.2}$$

where $\frac{dx^{\mu}}{d\lambda} = \dot{x}^{\mu}$ and λ is an affine parameter that satisfies the equation of the geodesic. Replacing the components from the metric tensor we get the following expression:

$$L = \frac{1}{2}\left[\left(1 - \frac{2\mu}{r}\right) c^2 \dot{t}^2 + \frac{4a\mu}{r} \dot{\phi} c \dot{t} - \frac{r^2}{a^2 - 2\mu \mathrm{r} + r^2} \dot{r}^2 - \left(a^2 + \frac{2a^2\mu}{r} + r^2\right) \dot{\phi}^2\right]. \tag{2.3}$$

Note that the langrangian is independent of t and ϕ, therefore

$$p_t = \frac{1}{c}\frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{2\mu}{r}\right) c\dot{t} + \frac{2a\mu}{r}\dot{\phi} = \frac{1}{c}E, \tag{2.4}$$

and

$$p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = \frac{2a\mu}{r} c\dot{t} - \left(a^2 + \frac{2a^2\mu}{r} + r^2\right)\dot{\phi} = -h, \tag{2.5}$$

are conserved quantities where E and h have units of energy per unit mass and angular momentum per unit mass, respectively.

Using (2.4) and (2.5), we get the equations of motion along the t and ϕ geodesics:

$$c\dot{t} = c\frac{dt}{d\lambda} = \frac{1}{\Delta}\left[\left(a^2 + \frac{2a^2\mu}{r} + r^2\right)\frac{E}{c} - \frac{2a\mu}{r}h\right], \tag{2.6}$$

$$\dot{\phi} = \frac{1}{\Delta}\left[\left(\frac{2a\mu}{r}\right)\frac{E}{c} + \left(1 - \frac{2\mu}{r}\right)h\right]. \tag{2.7}$$

Lastly, the expression for $\dot{r}$ is easily obtained from an alternative form of the Lagrangian [14, 6]:

$$g^{\mu\nu}p_\mu p_\nu = \eta^2, \tag{2.8}$$

where $p_\mu = \frac{dx_\mu}{d\lambda}$ is the covariant momentum, and $\eta = mc$ for massive particles or $\eta = 0$ for photons.

$$\implies g^{\mathrm{tt}}{p_t}^2 + 2g^{t\phi}p_t p_\phi + g^{\mathrm{rr}}{p_r}^2 + g^{\theta\theta}{p_\theta}^2 + g^{\phi\phi}{p_\phi}^2 = \eta^2. \tag{2.9}$$

Considering that

$$p_r = \frac{\partial L}{\partial \dot{r}} = -\frac{r^2}{\Delta}\dot{r}, \tag{2.10}$$

and $p_\theta = 0$, the corresponding values of the momentum can be replaced in equation 2.9 to get an expression for the r coordinate:

$$\begin{gathered}\dot{r}^2 = \left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{E'}{c}\right)^2 - \eta^2 + \frac{1}{r}\left(2\mu\eta^2\right) \\ + \frac{1}{r^2}\left(\left(a\frac{E'}{c}\right)^2 - (h')^2 - a^2\eta^2\right) + \frac{1}{r^3}2\mu\left(h' - a\frac{E'}{c}\right)^2\end{gathered} \tag{2.11}$$

With equations (2.11),(2.7) and (2.6) the motion of massive and massless particles in the equatorial plane is fully described.

2.2 Equatorial trajectories of photons

In this section we start the study of massless particles. First, we set $\eta = 0$ in equation (2.8), which results in the radial differential equation. (2.11) takes the following form:

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{E}{c}\right)^2 + \frac{1}{r^2}\left(\left(a\frac{E}{c}\right)^2 - h^2\right) + \frac{1}{r^3}2\mu\left(h - a\frac{E}{c}\right)^2. \tag{2.12}$$

Here E already has units of energy per unit mass, and h angular momentum per unit mass; λ is an affine parameter that satisfies the equation for null geodesics. The angular and temporal equations (2.7) and (2.6) are:

$$c\dot{t} = c\frac{dt}{d\lambda} = \frac{1}{\Delta}\left[\left(a^2 + \frac{2a^2\mu}{r} + r^2\right)\frac{E}{c} - \frac{2a\mu}{r}h\right], \tag{2.13}$$

$$\dot{\phi} = \frac{d\phi}{d\lambda} = \frac{1}{\Delta}\left[\left(\frac{2a\mu}{r}\right)\frac{E}{c} + \left(1 - \frac{2\mu}{r}\right)h\right]. \tag{2.14}$$

To simplify these equations we define a parameter $b \equiv \frac{hc}{E}$, so the equations of motion can be rewritten as follows:

$$\frac{1}{h^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} + \frac{1}{r^2}\left(\left(\frac{a}{b}\right)^2 - 1\right) + \frac{2\mu}{r^3}\left(1 - \frac{a}{b}\right)^2 \tag{2.15}$$

$$\frac{d\phi}{d\lambda} = \frac{h}{\Delta}\left[\left(\frac{2\mu}{r}\right)\frac{a}{b} + \left(1 - \frac{2\mu}{r}\right)\right] \tag{2.16}$$

Considering that $\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{dr}{d\phi}\right)^2 \left(\frac{d\phi}{d\lambda}\right)^2$, we find the equation of motion for a photon in the equatorial plane:

$$\Longrightarrow \frac{d\phi}{dr} = \frac{\left[\left(\frac{2\mu}{r}\right)\frac{a}{b} + \left(1 - \frac{2\mu}{r}\right)\right]}{r^2\left(1 - \frac{2\mu}{r} + \left(\frac{a}{r}\right)^2\right)\sqrt{\frac{1}{b^2} + \frac{1}{r^2}\left(\left(\frac{a}{b}\right)^2 - 1\right) + \frac{2\mu}{r^3}\left(1 - \frac{a}{b}\right)^2}} \tag{2.17}$$

Now, let us consider the case where a photon approaches a Kerr black hole from infinity, deviates because of the gravitational field and returns to infinity. Taking the limit $r \rightarrow \infty$ for equation 2.17, we obtain:

$$r^2 \frac{d\phi}{dr} = -b. \tag{2.18}$$

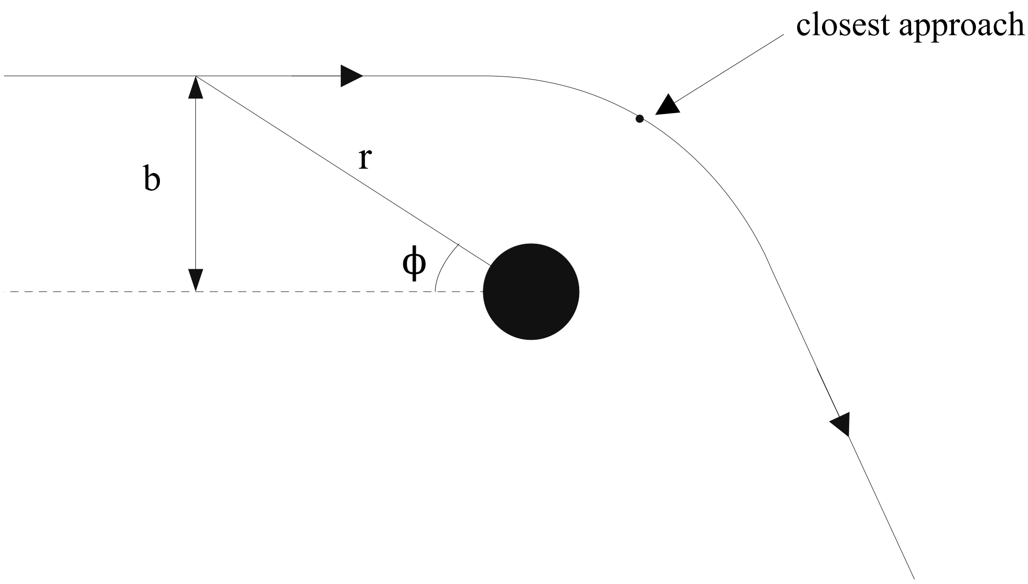


Figure 2.1: Impact parameter. (the distance of maximum approach is approximately b)

Figure 2.1 shows an example of the trajectory of a deflected photon by a black hole from infinity to infinity, where $\sin\phi = \frac{b}{r}$. By taking the derivative with respect to ϕ, we obtain

$d\phi \cos\phi = -\frac{b}{r^2}dr$, from where an approximation of small angle ϕ ($\cos\phi \approx 1$), equation (2.18) reveals that $r^2\frac{d\phi}{dr} = -b$. Thus proving that our defined parameter $b = \frac{hc}{E}$ is in fact, the impact parameter, or approximately the distance of maximum approach to the black hole.

Chapter III

Solving the equation of the orbit

3.1 Equation of the orbit

The equations for equatorial orbits in the Kerr metric (2.15) and (2.16) can be further simplified by applying a very well known transformation: $u(\phi) = \frac{1}{r(\phi)}$. Given that $\left(\frac{dr}{d\lambda}\right) = \left(\frac{dr}{d\phi}\right)\left(\frac{d\phi}{d\lambda}\right)$, we obtain the equation of the orbit:

$$\frac{1}{u^4}\left(\frac{du}{d\phi}\right)^2\left[(2\mu u)\frac{a}{b}+1-2\mu u\right]^2 = \frac{\Delta^2}{b^2} - \Delta^2 u^2\left(1-\frac{a^2}{b^2}-2\mu u\left(1-\frac{a}{b}\right)^2\right). \quad (3.1)$$

Now, we will approximate equation (3.1) to first order in $a = \mu s$, in an attempt to get a solvable differential equation:

$$\left(\frac{du}{d\phi}\right)^2 \sigma(u) = (1-2\mu u)\left[\frac{1}{b^2} - u^2\sigma(u)\right], \quad (3.2)$$

where $\sigma(u) \equiv 1-2\mu u+4\mu u\frac{a}{b}$. Deriving equation (3.2) with respect to ϕ:

$$\frac{d^2u}{d\phi^2}+\frac{\mu(1-2\mu u)}{\sigma^2}\left[\frac{1}{b^2}-u^2\sigma\right]\left(\frac{2a}{b}-1\right)=-\mu\left(\frac{1}{b^2\sigma}-u^2\right)$$
$$-(1-2\mu u)u\left[1+\frac{\mu u}{\sigma}\left(\frac{2a}{b}-1\right)\right] \tag{3.3}$$

Considering μu small and the first order in spin:

$$\begin{aligned}\sigma^{-1}(u) &\approx 1+2\mu u\left(1-2\tfrac{a}{b}\right)+4\mu^2u^2\left(1-4\tfrac{a}{b}\right)\\ \sigma^{-2}(u) &\approx 1+4\mu u\left(1-2\tfrac{a}{b}\right)-4\mu^2u^2\left(1-4\tfrac{a}{b}\right).\end{aligned}$$

Finally, after replacing the relations above in equation (3.3) we arrive to a general expression for the orbit which can be solved by taking advantage of its polynomial nature:

$$\frac{d^2u}{d\phi^2}+u=-\frac{2\mu^2s}{b^3}-\frac{8\mu^3s}{b^3}u+3\mu u^2-8\frac{\mu^3}{b^2}u^2\left(2-9\frac{\mu s}{b}\right)+8\frac{\mu^4}{b^2}u^3\left(1-6\frac{\mu s}{b}\right)$$
$$+24\mu^3u^4\left(1-6\frac{\mu s}{b}\right)-16\mu^4u^5\left(4-25\frac{\mu s}{b}\right)+16\mu^5u^6\left(1-8\frac{\mu s}{b}\right). \tag{3.4}$$

To solve equation (3.4) we will take a perturbative approach, for this, a small parameter must be defined. The critical radius of the orbit is of aid in this matter, and it can be easily studied by analyzing the specific case of circular trajectories on equation (3.4), taking only the first four terms of the right hand side of said equation:

$$\frac{d^2u}{d\phi^2}+u=3\mu u^2\left(1-\frac{16}{3}\left(\frac{\mu}{b}\right)^2\right)-\frac{2\mu^2s}{b^3}\left(1+4\mu u\right). \tag{3.5}$$

Defining $\delta\equiv\left(1-\frac{16}{3}\left(\frac{\mu}{b}\right)^2\right)$ and considering that $\frac{d^2u}{d\phi^2}=0$ for circular orbits, we end up with a quadratic equation which can be easily solved:

$$u_c = \frac{\left(1+\frac{8\mu^3 s}{b^3}\right)+\left[\left(1+\frac{8\mu^3 s}{b^3}\right)^2+\frac{24\mu^3 s}{b^3}\delta\right]^{1/2}}{6\mu\delta}, \tag{3.6}$$

considering that $\frac{\mu}{b}$ is small:

$$u_c \approx \frac{1}{3\mu\delta}. \tag{3.7}$$

The critical radius is will be approximately $r_c = 3\mu\delta < 3\mu$, given that $\delta < 1$ and the difference between 3μ and $3\mu\delta$ is negligible, therefore, we can define a small parameter $\varepsilon \equiv \frac{r_c}{b} = \frac{3\mu}{b}$ (a non-dimensional small number) with which, equation (3.4) can be rewritten. This will be done on the next section.

3.2 Perturbation theory

The equation for a photon orbiting a black hole in the Kerr metric is given by (3.4), which has a polynomial nature. In a first attempt to solve this equation to find the angle of deviation for a photon that approaches from infinity, we are going to try a perturbative treatment by Taylor series expansion. This can be done by expressing the equation of the orbit in terms of some small ε, and a function that converges when the photon effectively escapes (returns to infinity, see figure 3.1).

Let us define a converging power series :

$$V(\phi) = V_0(\phi) + \varepsilon V_1(\phi) + \varepsilon^2 V_2(\phi) + \varepsilon^3 V_3(\phi) + \varepsilon^4 V_4(\phi) + \dots, \tag{3.8}$$

which satisfies the following conditions:

$$\begin{aligned} V(\phi = 0) &= 1 \\ \frac{dV(\phi=0)}{d\phi} &= 0 \\ |V(\phi)| &\leq 1, \end{aligned}$$

where $V(\phi) \equiv \frac{b}{r(\phi)} = bu(\phi)$. Along with the small parameter $\varepsilon \equiv \frac{r_c}{b} = \frac{3\mu}{b}$, the equation of the orbit (3.4) can be rewritten as follows (up to third order in ε):

$$\frac{d^2V}{d\phi^2} + V = \varepsilon V^2 - \frac{2}{9}\varepsilon^2 s - \frac{8}{27}\varepsilon^3 sV - \frac{16}{27}\varepsilon^3 V^2 + \frac{8}{9}\varepsilon^3 V^4 \tag{3.9}$$

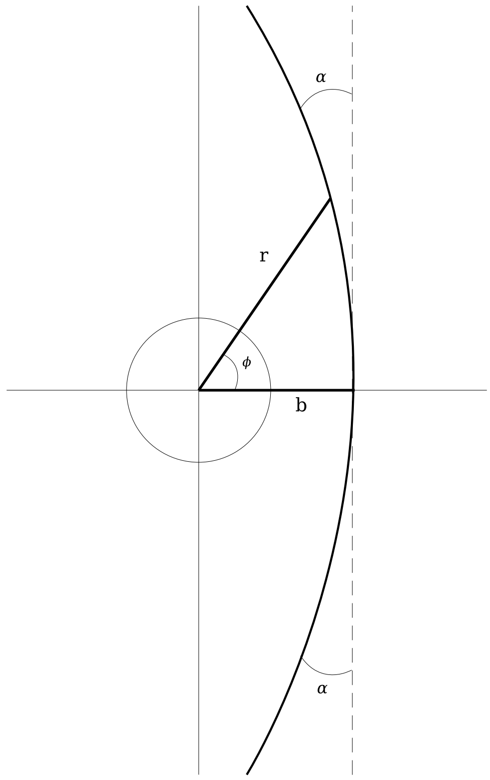


Figure 3.1: Angle of deviation 2α, where b is the impact parameter.

The trajectory of a photon that deviates due to the gravitational field of a Kerr black hole is depicted in figure (3.1), and it can be described using equation (3.9). Note that the spin parameter s appears in equation (3.9) in the second order term (ε^2), thus, at least third order series expansion would be necessary to accurately approximate the angle of deviation once we solve this equation. As a first attempt to find a solution, we will expand (3.9) as shown in equation (3.8):

$$\begin{aligned}
&\left(\frac{d^2V_0(\phi)}{d\phi^2}+\varepsilon\frac{d^2V_1(\phi)}{d\phi^2}+\varepsilon^2\frac{d^2V_2(\phi)}{d\phi^2}+\varepsilon^3\frac{d^2V_3(\phi)}{d\phi^2}+\dots\right)+\\
&+\left(V_0(\phi)+\varepsilon V_1(\phi)+\varepsilon^2V_2(\phi)+\varepsilon^3V_3(\phi)+\dots\right)=\\
&\ \varepsilon\ \left(V_0(\phi)+\varepsilon V_1(\phi)+\varepsilon^2V_2(\phi)+\varepsilon^3V_3(\phi)+\dots\right)^2-\frac{2}{9}\varepsilon^2 s\\
&-\frac{8}{27}\varepsilon^3 s\left(V_0(\phi)+\varepsilon V_1(\phi)+\varepsilon^2V_2(\phi)+\varepsilon^3V_3(\phi)+\dots\right)\\
&-\frac{16}{27}\varepsilon^3\left(V_0(\phi)+\varepsilon V_1(\phi)+\varepsilon^2V_2(\phi)+\varepsilon^3V_3(\phi)+\dots\right)^2\\
&+\frac{8}{9}\varepsilon^3\left(V_0(\phi)+\varepsilon V_1(\phi)+\varepsilon^2V_2(\phi)+\varepsilon^3V_3(\phi)+\dots\right)^4
\end{aligned} \tag{3.10}$$

Now that we have expanded the equation (3.9) in a power series, it is possible to solve equation (3.10) by separating it to each order of ε. This will lead to a system of equations that allows us to iteratively construct a solution for the function $V(\phi)$ of any desired order. Up to second order we have the following equations:

$$\begin{aligned}
&\text{For } \varepsilon^0: \quad \tfrac{d^2V_0}{d\phi^2}+V_0=0\\
&\text{For } \varepsilon^1: \quad \tfrac{d^2V_1}{d\phi^2}+V_1={V_0}^2\\
&\text{For } \varepsilon^2: \quad \tfrac{d^2V_2}{d\phi^2}+V_2=-\tfrac{2}{9}s+2V_0V_1
\end{aligned}$$

The initial conditions were given when the function $V(\phi)$ was defined:

$$\begin{aligned}
&\text{for } V_0: \quad V_0(\phi=0)=1, \quad \tfrac{dV_0(\phi=0)}{d\phi}=0,\\
&\text{for } V_i: \quad V_i(\phi=0)=0, \quad \tfrac{dV_i(\phi=0)}{d\phi}=0,\\
&\text{where} \quad i\in\mathbb{N}\geq 1.
\end{aligned}$$

Note that the condition changes for higher order terms of $V(\phi)$, as a consequence of the initial conditions of $V_0(\phi)$. The solutions are the following:

$$V_0(\phi)=\cos(\phi) \tag{3.11}$$

$$V_1(\phi)=\frac{1}{6}(3-2\cos(\phi)-\cos(2\phi)) \tag{3.12}$$

$$V_2(\phi)=-\frac{1}{3}-\frac{2}{9}s+\left(\frac{29}{144}+\frac{2}{9}s\right)\cos(\phi)+\frac{1}{9}\cos(2\phi)+\frac{1}{48}\cos(3\phi)+\frac{5}{12}\phi\sin(\phi). \tag{3.13}$$

Before attempting to find the angle of deviation, let's look at $V_2(\phi)$. The second order equation contains a term which misbehaves in a series such as this one ($\frac{5}{12}\phi\sin(\phi)$), it grows indefinitely with ϕ. This happens because the homogeneous solution to the second order equation (ε^2) $a\cos(\phi)+b\sin(\phi)$ contains terms proportional the right hand side of said equation. Terms such as $\phi\sin(\phi)$ are called *secular terms*, if we naively include these terms in the solution $V(\phi)$, it will no longer be bounded. This is of course a problem, because the initial conditions with which the $V(\phi)$ was defined would not be met. One method to eliminate this secular terms and obtain a well behaved solution is the Lindstedt-Poincaré method, as we will see in the next section. Nonetheless, we shall calculate the angle of deviation to second order anyway. The function $V(\phi)=V_0(\phi)+\varepsilon V_1(\phi)+\varepsilon^2 V_2(\phi)$ can be put together as such:

$$V(\phi) = \cos\phi + \varepsilon\left(\frac{1}{6}(3 - 2\cos(\phi) - \cos(2\phi))\right)$$
$$+\varepsilon^2\left(-\frac{1}{3} - \frac{2}{9}s + \left(\frac{29}{144} + \frac{2}{9}s\right)\cos(\phi) + \frac{1}{9}\cos(2\phi) + \frac{1}{48}\cos(3\phi) + \frac{5}{12}\phi\sin(\phi)\right), \tag{3.14}$$

remember that $V(\phi) = \frac{b}{r}$, the following condition must be satisfied:

$$\text{as } r \to \infty \Rightarrow V \to 0.$$

Therefore, when a photon is deviated, there must be an angle α that satisfies $V\left(\frac{\pi}{2} + \alpha\right) = 0$. Replacing ϕ with $\frac{\pi}{2} + \alpha$ in equation 3.14 and solving for α gives the expression for the angle of deflection of light, once we eliminate all the higher order terms:

$$\alpha = \frac{2}{3}\varepsilon + \frac{\varepsilon^2}{9}\left(\frac{15\pi}{8} - 2(s+1)\right) \tag{3.15}$$

The total angle of deviation is $\Omega = 2\alpha$:

$$\Omega = \frac{4}{3}\varepsilon + \frac{\varepsilon^2}{9}\left(\frac{15\pi}{4} - 4(s+1)\right), \tag{3.16}$$

given that $\varepsilon = \frac{3\mu}{b} = \frac{3GM}{bc^2}$:

$$\Longrightarrow \Omega = \frac{4GM}{bc^2} + \left(\frac{GM}{bc^2}\right)^2\left(\frac{15\pi}{4} - 4(s+1)\right). \tag{3.17}$$

Setting $s = 0$, this result agrees with previous studies [15][16][17][18].

3.3 Lindstedt-Poincaré

We have successfully obtained the angle of deviation for a photon in the Kerr metric, to second order. This result is consistent with previous studies of second order corrections to the deflection angle [19][11]. The secular term that was mentioned previously does not affect the second order terms, it appears in third and higher orders. Therefore, to be able to calculate a third order solution we need to get rid of all secular terms that appear in the differential equations, to do this we employ the Lindstedt-Poincaré method. To eliminate the divergent terms from the higher order differential equations, an angle $\tilde{\phi}$ is defined as a power series in ε:

$$\tilde{\phi} = \phi \left(1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \omega_3 \varepsilon^3 + \ldots\right), \tag{3.18}$$

where ω_i is a parameter that eliminates the secular term in the corresponding i^{th} order equation. Rewriting 3.9 in terms of $\tilde{\phi}$, to third order:

$$\begin{aligned}\left(1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \omega_3 \varepsilon^3\right)^2 \frac{d^2 V\left(\tilde{\phi}\right)}{d\tilde{\phi}^2} + V\left(\tilde{\phi}\right) =& \varepsilon V^2\left(\tilde{\phi}\right) - \frac{2}{9}\varepsilon^2 s - \frac{8}{27}\varepsilon^3 s\, V\left(\tilde{\phi}\right) \\ & - \frac{16}{27}\varepsilon^3 V^2\left(\tilde{\phi}\right) + \frac{8}{9}\varepsilon^3 V^4\left(\tilde{\phi}\right).\end{aligned} \tag{3.19}$$

Performing the expansion $V(\tilde{\phi}, \varepsilon) = V_0(\tilde{\phi}) + \varepsilon V_1(\tilde{\phi}) + \varepsilon^2 V_2(\tilde{\phi}) + \varepsilon^3 V_3(\tilde{\phi}) + \ldots$ in the equation above, produces the following system of equations:

$$\begin{aligned}
\text{For } \varepsilon^0: \quad & \tfrac{d^2V_0}{d\phi^2}+V_0 &&= 0\\
\text{For } \varepsilon^1: \quad & \tfrac{d^2V_1}{d\tilde{\phi}^2}+V_1 &&= {V_0}^2-2\omega_1 V_0\text{''}\\
\text{For } \varepsilon^2: \quad & \tfrac{d^2V_2}{d\tilde{\phi}^2}+V_2 &&= -\tfrac{2}{9}s+2V_0V_1-\left({\omega_1}^2+2\omega_2\right)V_0\text{''}-2\omega_1 V_1\text{''}\\
\text{For } \varepsilon^3: \quad & \tfrac{d^2V_3}{d\tilde{\phi}^2}+V_3 &&= -\tfrac{8s}{27}V_0-\tfrac{16}{27}{V_0}^2+\tfrac{8}{9}{V_0}^4+{V_1}^2+2V_0V_2-(2\omega_1\omega_2+2\omega_3)V_0''\\
& && \quad -\left(\omega_1^2+2\omega_2\right)V_1''-2\omega_1 V_2''
\end{aligned}$$

To solve the equations above, the same initial conditions have to be considered:

$$\begin{aligned}
&\text{for } V_0: \quad V_0(\phi=0)=1 \quad \tfrac{dV_0(\phi=0)}{d\phi}=0,\\
&\text{for } V_i: \quad V_i(\phi=0)=0 \quad \tfrac{dV_i(\phi=0)}{d\phi}=0,\\
&\text{where} \quad i\in\mathbb{N}\geq 1,
\end{aligned}$$

before arriving at the solutions, the values of ω_i have to be determined so the secular terms are eliminated.

$$\begin{aligned}
\omega_1 &= 0\\
\omega_2 &= -\tfrac{5}{12}\\
\omega_3 &= \tfrac{1}{54}(15+20s)
\end{aligned}$$

Introducing these parameters into the solutions to the differential equations, we obtain well behaved solutions with the divergent terms removed:

$$
\begin{aligned}
V_0\left(\tilde{\phi}\right) &= \cos(\phi)\\
V_1\left(\tilde{\phi}\right) &= \tfrac{1}{6}(3-2\cos(\phi)-\cos(2\phi))\\
V_2\left(\tilde{\phi}\right) &= \tfrac{1}{144}\left(-48-32s+29\cos(\phi)+32s\cos(\phi)+16\cos(2\phi)+3\cos(3\phi)\right)\\
V_3\left(\tilde{\phi}\right) &= \tfrac{1}{6480}\left(3615+1440s-1657\cos(\phi)-960s\cos(\phi)-1760\cos(2\phi)-480s\cos(2\phi)\right.\\
&\quad\left.-135\cos(3\phi)-63\cos(4\phi)\right)
\end{aligned}
$$

The solution is $V(\tilde{\phi},\varepsilon)=V_0(\tilde{\phi})+\varepsilon V_1(\tilde{\phi})+\varepsilon^2 V_2(\tilde{\phi})+\varepsilon^3 V_3(\tilde{\phi})$, to third order:

$$
\begin{aligned}
V(\tilde{\phi},\varepsilon) = {}& \cos(\tilde{\phi})+\frac{\varepsilon}{6}(3-2\cos(\tilde{\phi})-\cos(2\tilde{\phi}))+\frac{\varepsilon^2}{144}\left(-48-32s+29\cos(\tilde{\phi})+32s\cos(\tilde{\phi})\right.\\
&\left.+16\cos(2\tilde{\phi})+3\cos(3\tilde{\phi})\right)+\frac{\varepsilon^3}{6480}\left(3615+1440s-1657\cos(\tilde{\phi})-960s\cos(\tilde{\phi})\right.\\
&\left.-1760\cos(2\tilde{\phi})-480s\cos(2\tilde{\phi})-135\cos(3\tilde{\phi})-63\cos(4\tilde{\phi})\right)
\end{aligned}
\tag{3.20}
$$

In an attempt to simplify the previous equation, we rewrite it in terms of powers of cosine:

$$
\begin{aligned}
V(\tilde{\phi},\varepsilon) = {}& \cos\tilde{\phi}+\frac{1}{3}\left(2-\cos(\tilde{\phi})-\cos^2(\tilde{\phi})\right)\varepsilon+\frac{1}{36}\left(-16-8s+5\cos(\tilde{\phi})+8s\cos(\tilde{\phi})\right.\\
&\left.+8\cos^2(\tilde{\phi})+3\cos^3(\tilde{\phi})\right)\varepsilon^2+\left(\frac{332}{405}+\frac{8s}{27}-\frac{313}{1620}\cos(\tilde{\phi})-\frac{4s}{27}\cos(\tilde{\phi})\right.\\
&\left.-\frac{377}{810}\cos^2(\tilde{\phi})-\frac{4s}{27}\cos^2(\tilde{\phi})-\frac{1}{12}\cos^3(\tilde{\phi})-\frac{7}{90}\cos^4(\tilde{\phi})\right)\varepsilon^3.
\end{aligned}
\tag{3.21}
$$

The solution to the equation of motion allows us to find the angle of deviation of a deflected photon (remember that the condition $V(\frac{\pi}{2}+\tilde{\alpha})=0$ must be satisfied). As it can be observed in equation (3.21), we would need to solve an equation of polynomial nature with a sine function of increasing degree; evidently, this becomes very troublesome to deal with at higher orders. Therefore, the $\sin(\alpha)$ function can be expressed in terms of a power series with ε as a leading term (this allows the sine function to behave properly at small angles):

$$\Longrightarrow \sin(\tilde{\alpha}) = \varepsilon\chi_1 + \varepsilon^2\chi_2 + \varepsilon^3\chi_3 + \dots \tag{3.22}$$

First, replacing $\tilde{\phi} = \frac{\pi}{2} + \tilde{\alpha}$, we get the following equation:

$$\begin{aligned}0 = &-\sin\tilde{\alpha} + \frac{\varepsilon}{3}\left(2+\sin\tilde{\alpha}-\sin^2\tilde{\alpha}\right) + \frac{\varepsilon^2}{36}(-16-8s-5\sin\tilde{\alpha}-8s\sin\tilde{\alpha} \\ &+8\sin^2\tilde{\alpha}-3\sin^3\tilde{\alpha}) + \varepsilon^3\left(\frac{332}{405}+\frac{8s}{27}+\frac{313}{1620}\sin\tilde{\alpha}+\frac{4s}{27}\sin\tilde{\alpha}-\frac{377}{810}\sin^2\tilde{\alpha}\right. \\ &\left.-\frac{4s}{27}\sin^2\tilde{\alpha}+\frac{1}{12}\sin^3\tilde{\alpha}-\frac{7}{90}\sin^4\tilde{\alpha}\right)\end{aligned} \tag{3.23}$$

Now, applying the series expansion of the sine function, we can find the χ_i coefficients, which construct the angle of deviation.

$$\begin{aligned}0 = &\left(\frac{2}{3}-\chi_1\right)\varepsilon + \frac{1}{9}(-4-2s+3\chi_1-9\chi_2)\varepsilon^2 + \frac{1}{1620}(1328+480s-225\chi_1 \\ &-360s\chi_1-540\chi_1^2+540\chi_2-1620\chi_3)\varepsilon^3\end{aligned} \tag{3.24}$$

$$\begin{aligned}\Longrightarrow \quad &\chi_1 \to \tfrac{2}{3} \\ &\chi_2 \to -\tfrac{2}{9}(1+s) \\ &\chi_3 \to \tfrac{1}{810}(409+60s)\end{aligned}$$

$$\Longrightarrow \sin\tilde{\alpha} = \frac{2\varepsilon}{3} - \frac{2}{9}(1+s)\varepsilon^2 + \frac{1}{810}(409+60s)\varepsilon^3 + \dots \tag{3.25}$$

To find $\tilde{\alpha}$, we need to apply the inverse sine function (this of course has to be expanded in its Taylor series to the desired order). Since we are working up to third order, the expansion is as follows:

$$\arcsin x = x + \frac{x^3}{6} + O[x]^5$$

$$\Longrightarrow \tilde{\alpha} = \frac{2\varepsilon}{3} - \frac{2}{9}(1+s)\varepsilon^2 + \left(\frac{449}{810} + \frac{2s}{27}\right)\varepsilon^3 \tag{3.26}$$

Remember that the Lindstedt-Poincaré method expands the angle as a power series, thus, to find the actual deviation angle we need to revert the transformation.

$$\tilde{\phi} = \phi\left(1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \omega_3\varepsilon^3\right)$$

$$\Longrightarrow \frac{\pi}{2} + \tilde{\alpha} = \left(\frac{\pi}{2} + \alpha\right)\left(1 - \frac{5}{12}\varepsilon^2 + \frac{1}{54}(15+20s)\varepsilon^3\right)$$

$$\frac{\pi}{2} + \left(\frac{2\varepsilon}{3} - \frac{2}{9}(1+s)\varepsilon^2 + \left(\frac{449}{810} + \frac{2s}{27}\right)\varepsilon^3\right) = \left(\frac{\pi}{2} + \alpha\right)\left(1 - \frac{5}{12}\varepsilon^2 + \frac{1}{54}(15+20s)\varepsilon^3\right)$$

$$\Longrightarrow \alpha = \frac{2\varepsilon}{3} + \frac{1}{72}(-16+15\pi-16s)\varepsilon^2 + \frac{(1348-225\pi+120s-300\pi s)\varepsilon^3}{1620} \tag{3.27}$$

Finally, we can find the total deviation:

$$\Omega = 2\alpha = \frac{4\varepsilon}{3} + \frac{\varepsilon^2}{36}(-16 + 15\pi - 16s) + \frac{\varepsilon^3}{810}(1348 - 225\pi + 120s - 300\pi s)$$

$$\Omega = \frac{4\varepsilon}{3} + \frac{\varepsilon^2}{9}\left(\frac{15\pi}{4} - 4(1+s)\right) + \frac{\varepsilon^3}{27}\left(\frac{674}{15} - \frac{15\pi}{2} + (4 - 10\pi)s\right) \tag{3.28}$$

Observe that the first two terms in 3.28 are in agreement with the ones given by equation 3.17.

3.4 Results

The figures in this section show the deviation angle and compare it to the numerical values. It is clear that the perturbative approach conserves the tendency of the function which gives the angle of deviation as a function of spin and the impact parameter, nonetheless, the approximation could be more precise. In the next section we are going to explore the method of Padé approximation, applied to the already calculated angle of deviation in an attempt to increase the precision of our results. The numerical values come from solving equation (3.3) numerically for each value of spin and impact parameter.

Figure 3.2 shows the numeric solution to the general equation of the orbit 3.3 for the spinless case, compared to the solution found in equation 3.19. This result clearly shows that the perturbative solution works only for small values of ε. Next, figure 3.3 shows the behavior of a photon's deviation angle as it approaches a rotating black hole at different distances from its center, note that the deviation angle strongly depends on the spin parameter that the black hole holds. The solutions from the Lindstedt-Poincaré method, represented in figure 3.3, are compared to the numeric solution to equation 3.3. Note that the error of the perturbative method increases as the particle approaches the black hole, specially for figures (3.3c) and (3.3d). This proves that higher orders of the Lindstedt-Poincaré solutions are necessary to accurately describe the behavior of light's deviation near a rotating black hole. Finally, on all figures the y axis represents the deviation angle, and the x axis ε (which tells us how close to the black hole the particle passes).

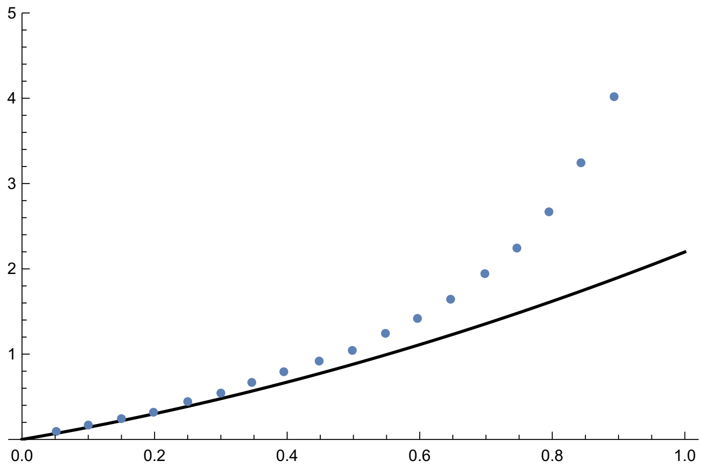
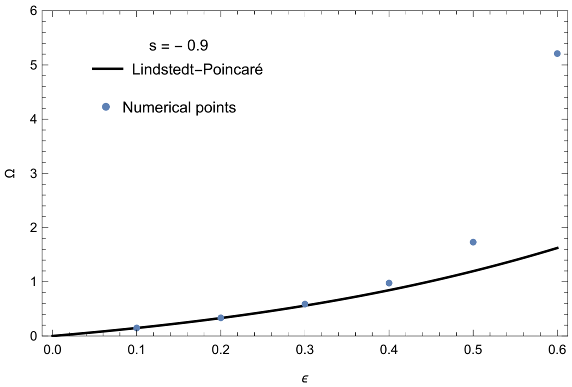
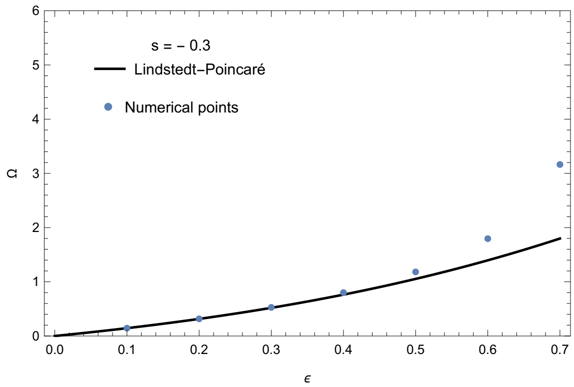


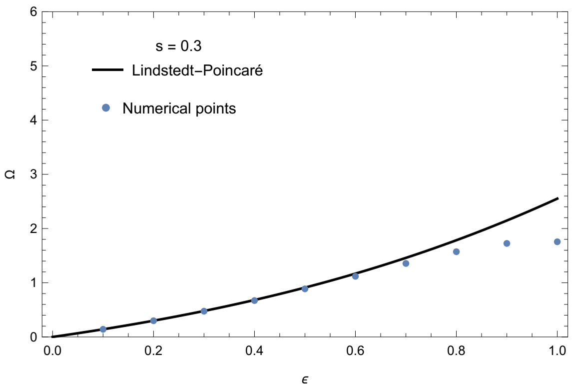
Figure 3.2: Numerical solution compared to the second order Taylor solution in equation (3.17), for the spinless case.



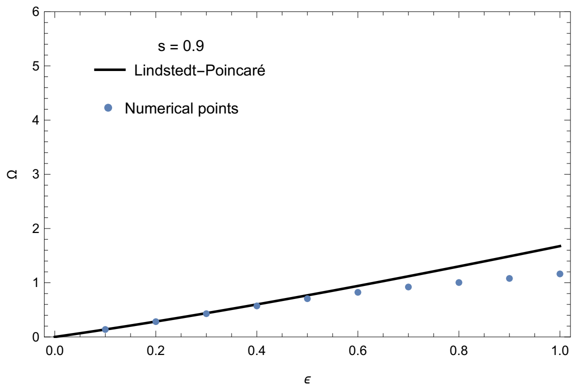
(a)



(b)



(c)



(d)

Figure 3.3: Angle of deviation as a function of ε for different spin parameters. The solid line represents the solution given by equation (3.21), and the points are the solutions to equation (3.3). These plots show the Lindstedt-Poincaré and numerical solutions for spin parameters: (a) $s = -0.9$, (b) $s = -0.3$, (c) $s = 0.3$ and (d) $s = 0.9$.

3.5 Padé Approximants

The method of Padé [20] will be employed to find a rational approximation of the deviation angle, which was calculated as a power series. This method has been used to study the light deviation near Schwarzschild and Reissner-Nordstrom black holes [11][19], and also in Cosmology. The Padé approximant is defined as follows:

Given a power series:

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

The rational function of order $[m/n]$:

$$R^{[m/n]}(x) = \frac{a_0 + a_1 x + \ldots + a_m x^m}{1 + b_1 x + \ldots + b_n x^n},$$

must match the power series $f(x)$ up to its derivative of order $m+n$:

$$R(0) = f(0), R'(0) = f'(0), \ldots, R^{(m+n)}(0) = f^{(m+n)}(0).$$

To find the coefficients of the polynomials in $R^{[m/n]}$, the following system of equations is used [20], which satisfies the conditions stated above.

$$
\begin{aligned}
&c_0 b_0 = a_0,\\
&c_1 b_0 + c_0 b_1 = a_1,\\
&\dots\\
&c_m b_0 + c_{m-1} b_1 + \ldots + c_{m-n} b_n = a_m,\\
&c_{m+1} b_0 + c_m b_1 + \ldots + c_{m-n+1} b_n = 0,\\
&\dots\\
&c_{m+n} b_0 + c_{m+n-1} b_1 + \ldots + c_m b_n = 0.\\
&(\text{If } i < 0 \Rightarrow c_i = 0)
\end{aligned}
$$

For the deviation angle calculated with the Lindstedt-Poincaré method (3.28), applying the procedure above, we obtain the following Padé approximants:

$$
\begin{aligned}
\Omega^{[1/1]}(\varepsilon,s) &\equiv R^{[1/1]}(\varepsilon,s) = \frac{64\varepsilon}{48-15\pi\varepsilon+16(1+s)\varepsilon}\\
\Omega^{[1/2]}(\varepsilon,s) &\equiv R^{[1/2]}(\varepsilon,s) = \frac{46080\varepsilon}{34560+\varepsilon\left(3375\pi^2\varepsilon+1200\pi(-9+2s\varepsilon)+128(90-307\varepsilon+30s(3+\varepsilon+s\varepsilon))\right)}\\
\Omega^{[2/1]}(\varepsilon,s) &\equiv R^{[2/1]}(\varepsilon,s) = \frac{\varepsilon\left(3375\pi^2\varepsilon+1200\pi(9+2s\varepsilon)+128(-90-307\varepsilon+30s(-3+\varepsilon+s\varepsilon))\right)}{900\pi(9+(6+8s)\varepsilon)-96(90+337\varepsilon+30s(3+\varepsilon))}
\end{aligned}
$$

To compute higher order Padé approximants, it is necessary to calculate the angle of deviation with the Lindstedt-Poincaré method of order $n+m$. To accurately approximate the angle of deviation, it was found that at least a fifth order solution is necessary. The higher order solutions for the angle calculated with the Lindstedt-Poincaré and Padé methods are given in appendices B and C, respectively.

3.6 Analysis and Numerical Tests

The Padé approximants calculated numerically can be compared to the numerical solution of the equation, and to the results from the Lindstedt-Poincaré method. Clearly, the Padé approximants provide more accurate results, especially as ε increases, compared to the Lindstedt-Poincaré solution. This means that the rational approximation given by Padé provides means to find a better fit for the solution to equation (3.3) than equations (3.21) and (B.1). Now, determining which Padé approximant is best for each case is a manual process that involves calculating the statistical error between each of the numerical points and the corresponding Padé values, then taking the mean error. This allows us to choose the Padé approximant that fits the numerical points best, overall.

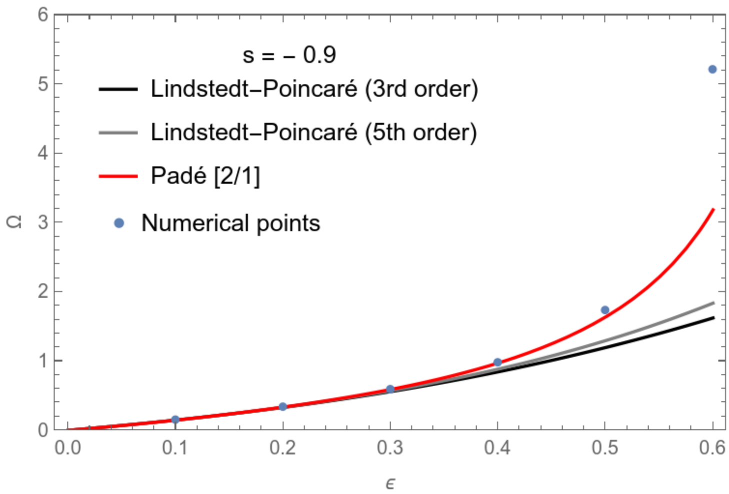


Figure 3.4: Angle of deviation as a function of ε for spin parameter: $s = -0.9$. The solid lines are given by equations (3.21), (B.1) and $\Omega^{[2/1]} = R^{[2/1]}$, compared to the numerical solution to equation (3.3). The statistical error for the Padé approximation is $e = 7.60\%$.

From figures (3.4) and (3.5) it can be observed that the error increases for higher spin values, also, when considering points near $\varepsilon = 1$. This is expected, since the perturbation method works for small values of ε, and the Padé approximant is derived from the Lindstedt-Poincaré solution, nonetheless, the Padé approximant for each case gives a reasonable approximation. The mean statistical error for the Padé approximants with respect to the numerical solutions is between $1.35\% - 6.48\%$ for different spin parameters and Padé approximants. Therefore, we have found expressions that correctly describe the behavior of photons deviating their trajectory due to the gravitational field of a rotating black hole.

Lastly we applied the studied method of approximation to the specific case of the binary black hole system OJ-287 in figure 3.6, considering that the main (most massive) black hole of the system has a spin parameter $s = 0.381$ [21], yielding very good results.

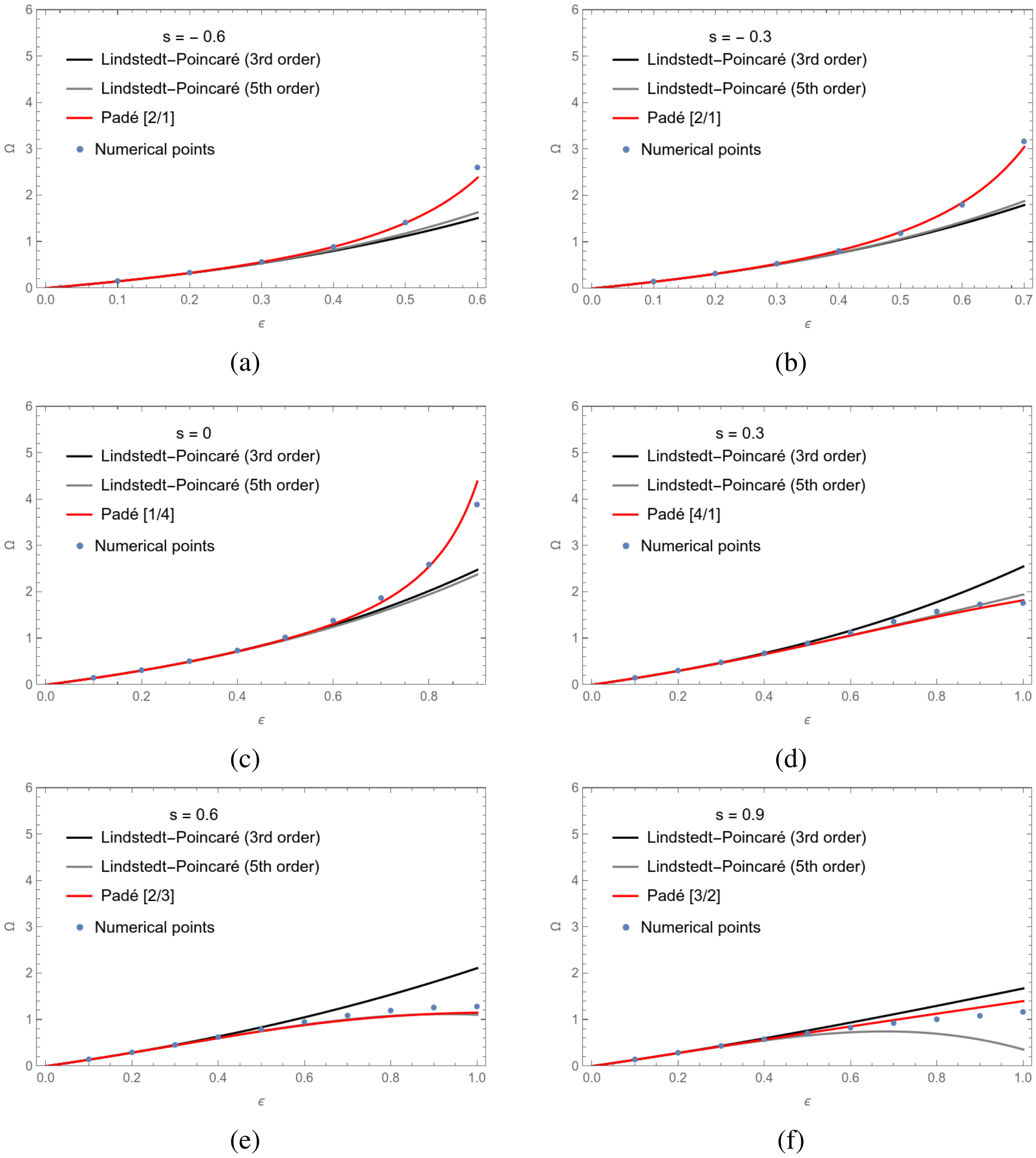
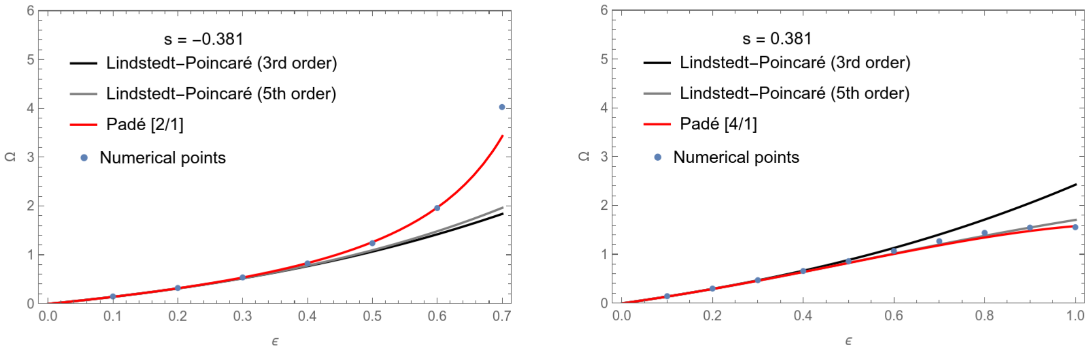


Figure 3.5: Angle of deviation as a function of ε for different spin parameters. Comparison of equations (3.21), (B.1), and different orders of Padé approximants with the numerical solution to equation (3.3). These plots consider the spin parameters and the mean statistical error for the Padé approximation: (a) $s = -0.6$, $e = 1.83\%$, (b) $s = -0.3$, $e = 2.02\%$, (c) $s = 0$, $e = 3.29\%$, (d) $s = 0.3$, $e = 3.24\%$, (e) $s = 0.6$, $e = 5.25\%$, and (f) $s = 0.9$, $e = 6.48\%$.



(a) (b)

Figure 3.6: Angle of deviation as a function of ε for different spin parameters for a photon passing near the OJ-287 system. Comparison of equations (3.21), (B.1), and different orders of Padé approximants with the numerical solution to equation (3.3). The retrograde orbit(a) and the direct orbit (b) are shown.

CONCLUSIONS

In this dissertation, we present a way to solve the equation of motion for null geodesics in the equatorial plane of the Kerr metric. Being particularly interested in one of the first and most exciting predictions of General Relativity, the deviation of light that passes in the vicinity of a strong gravitational field. We focus on rotating black holes, the total angular momentum carried by these enormous compact objects curves space-time in a very interesting way. The frame dragging effect that occurs around a Kerr black hole can indeed make the equations that describe the motion of particles, very complex; this is why we have simplified the metric to the equatorial plane. Before studying light deviation, we found two interesting results consequential to the Kerr metric. First, equations D.29 and D.30 give an analytical expression for the radius of circular orbits in Kerr space-time, given the parameters such as mass and spin. Secondly, in section 2.2.2 the calculation of released energy of a particle falling from infinity to the smallest possible circular orbit around a Kerr black hole. Table 4.2 shows how much of the rest energy of a particle is lost when falling to a circular orbit.

As a first attempt to solve the equation of motion, we try a traditional perturbative treatment, with a small parameter $\varepsilon = \frac{r_c}{b}$, but this method yields a problem for solutions of higher order than two. When trying to solve for third order, we find ourselves dealing with terms that grow boundlessly, this of course is unwanted behavior in our solution, convergence is necessary.

These secular terms of the form $\phi \sin \phi$ are oscillating with a growing amplitude, which may lead to non-uniformity in the solutions; additionally, difficulty in solving equations of order n arises. To go around this issue, we applied the Lindstedt-Poincaré method, which expands the variable that appears in the secular terms, this allows for adequate behavior once the coefficients are chosen correctly such that the secular term is eliminated. As it can be observed from the plots presented in section 3.1.1, this method preserves the behavior of the numerical solution, yet it lacks precision. Finally, in an attempt to further increase precision in the approximation of the deviation angle, Padé approximants were calculated from the result of the Lindstedt-Poincaré method. From the plots shown of the Padé expressions, we can see that they increase precision for the angle of deviation. In previous work [19][11], it was shown that the Padé approximants produce better results than Lindstedt-Poincaré in the Schwarzschild and Reissner-Nordstrom metrics, this is also the case for the Kerr metric. The plots presented in previous section show that the Lindstedt-Poincaré method will produce better results for small ε, which is expected from the perturbative nature of the solution. In order to approximate the angle for regions closer to the black hole, the Padé method was employed, producing favorable results. It may be possible to find better approximations with higher order solutions, to calculate the higher order terms for the Lindstedt-Poincaré method, a similar procedure as the one shown in section 3.3 can be followed, the same goes for the Padé approximants in section 3.5.

Finally, in Appendix A, equation 3.4 is solved numerically and the results of the orbits are plotted. Here, we can observe that said equation works very well for any ε, and is able to reproduce light deviation for extreme parameters such as $\varepsilon = 1$.

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Appendix

Appendix A

Photon trajectories near the Kerr black hole

In this appendix we are going to solve equation 3.4 numerically and plot the trajectories of massless particles which approach a rotating black hole at any distance. Thus, obtaining a graphical approach to the analysis that was done in this dissertation.

1.1 Retrograde orbits

For photons that approach the black hole in the direction opposite to its rotation.

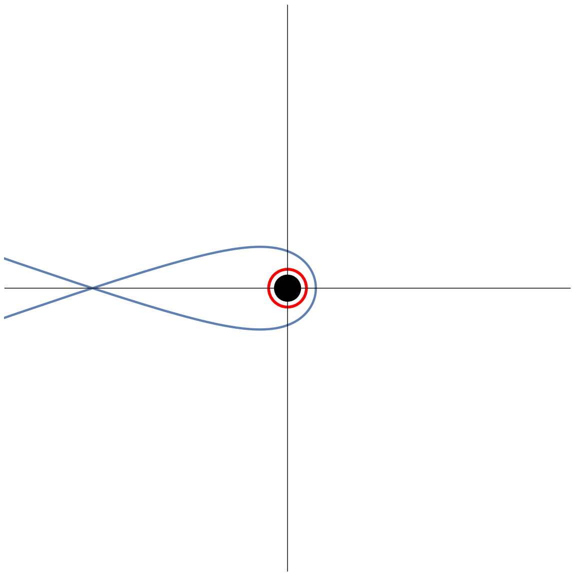

Figure 1.1: Massless particle escaping a Kerr black hole. $s \to -0.9$, $\varepsilon \to 1$

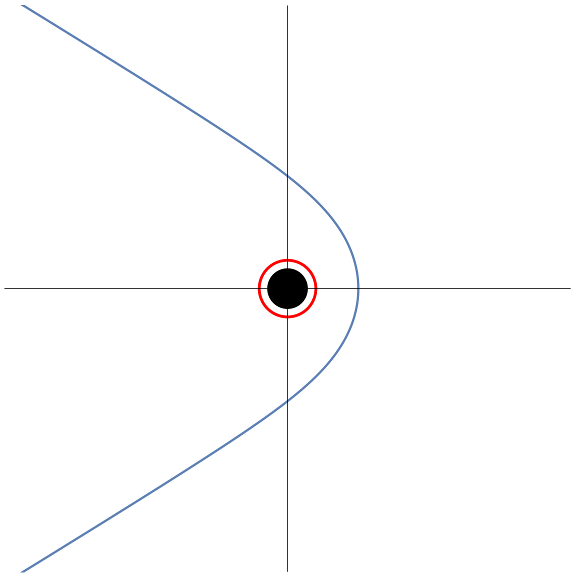

Figure 1.2: Massless particle escaping a Kerr black hole. $s \to -0.9$, $\varepsilon \to 0.6$

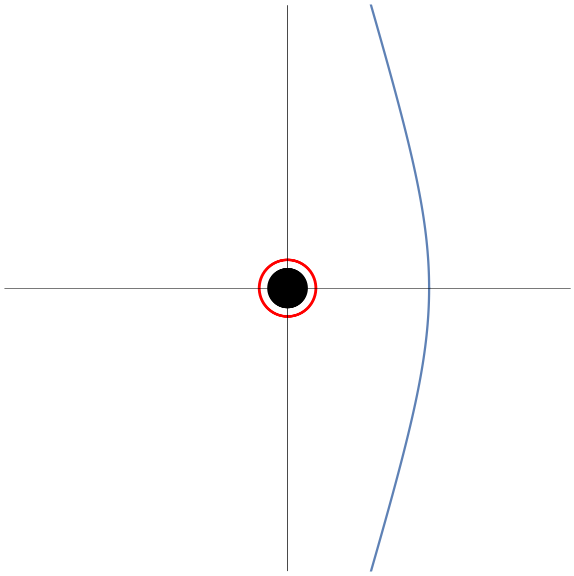

Figure 1.3: Massless particle escaping a Kerr black hole. $s \to -0.9$, $\varepsilon \to 0.3$

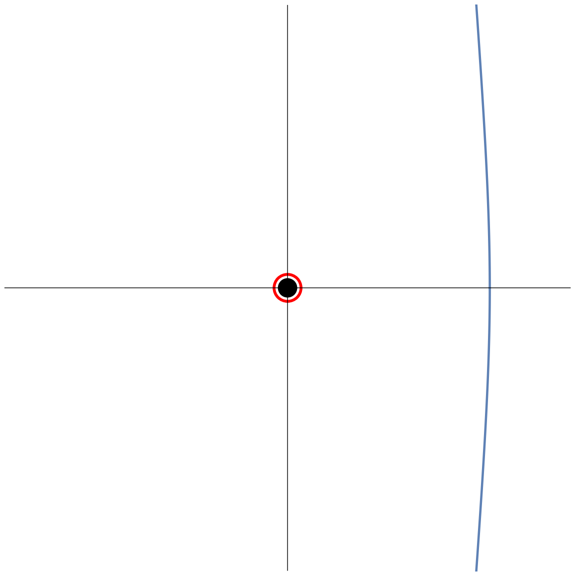

Figure 1.4: Massless particle escaping a Kerr black hole. $s \to -0.9$, $\varepsilon \to 0.1$

1.2 Direct orbits

For photons that approach the black hole in the same direction as its rotation.

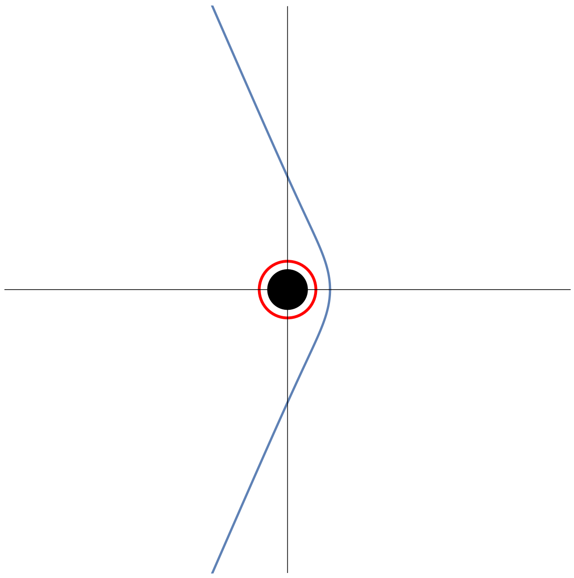

Figure 1.5: Massless particle escaping a Kerr black hole. $s \to 0.9$, $\varepsilon \to 1$

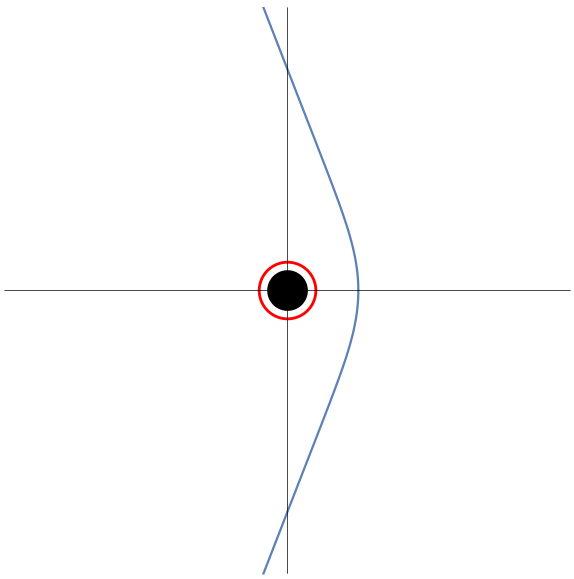

Figure 1.6: Massless particle escaping a Kerr black hole. $s \to 0.9$, $\varepsilon \to 0.6$

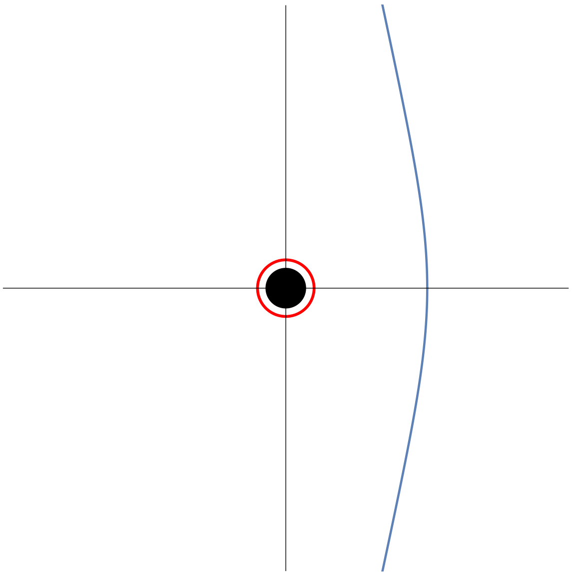

Figure 1.7: Massless particle escaping a Kerr black hole. $s \to 0.9$, $\varepsilon \to 0.3$

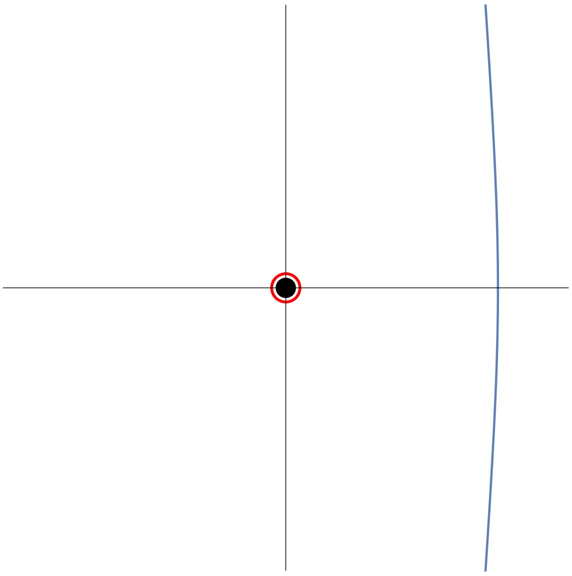

Figure 1.8: Massless particle escaping a Kerr black hole. $s \to 0.9$, $\varepsilon \to 0.1$

Appendix B

Higher order Lindstedt-Poincaré solutions for the angle of deviation

In this appendix we show the analytical solution to the tenth order Lindstedt-Poincaré equation. Note that to extract lower order solutions one only needs to take the terms up to the desired order in ε. It is important to mention that calculating solutions of order higher than 10, the analytical method is not practical, since it takes a lot of resources to continue solving n order equations analytically. It is recommended to use numerical methods for higher order solutions.

For example, the fifth order solution would be extracted from equation B.2 as such:

$$\begin{aligned}\Omega_5(\varepsilon,s) = {} & \frac{4\varepsilon}{3} + \frac{1}{36}(-16+15\pi-16s)\varepsilon^2 + \frac{1}{810}(1348-225\pi+120s-300\pi s)\varepsilon^3 \\ & + \frac{\left(-176000+44235\pi-191616s+14400\pi s+7680s^2\right)\varepsilon^4}{77760} \\ & + \frac{\left(1489396-427245\pi+1564192s-484680\pi s+161280s^2\right)\varepsilon^5}{408240}\end{aligned} \tag{B.1}$$

$$
\begin{aligned}
\Omega_{10}(\varepsilon,s) = {} & \frac{4\varepsilon}{3}+\frac{1}{36}(-16+15\pi-16s)\varepsilon^2+\frac{1}{810}(1348-225\pi+120s-300\pi s)\varepsilon^3 \\
& + \frac{\left(-176000+44235\pi-191616s+14400\pi s+7680s^2\right)\varepsilon^4}{77760} \\
& + \frac{\left(1489396-427245\pi+1564192s-484680\pi s+161280s^2\right)\varepsilon^5}{408240} \\
& + \frac{\varepsilon^6}{293932800}\big(-1921390528+443086035\pi-2596559040s+709480800\pi s \\
& - 368686080s^2+176198400\pi s^2-7526400s^3\big) \\
& + \frac{\varepsilon^7}{3086294400}\big(34127489924-8259932415\pi+55195454496s \\
& - 13348108620\pi s+19698470400s^2-4625913600\pi s^2-440294400s^3\big) \\
& + \frac{\varepsilon^8}{1185137049600}\big(-24310353374720+5519903716005\pi-43420385554944s \\
& + 10621758846720\pi s-18408529649664s^2+4825116334080\pi s^2 \\
& - 1485964247040s^3+312134860800\pi s^3+8670412800s^4\big) \\
& + \frac{\varepsilon^9}{571405363200}\big(21235920708674-4791043404171\pi+44666461143360s \\
& - 10331021594880\pi s+24564817893888s^2-5990765598720\pi s^2 \\
& + 2450576056320s^3-702709862400\pi s^3+29340057600s^4\big) \\
& + \frac{\varepsilon^{10}}{639974006784000}\big(-44121927235224128+9718950366662355\pi \\
& - 102625235958870080s+23450778052873920\pi s-70736680982016000s^2 \\
& + 16806020482944000\pi s^2-12546414990950400s^3+2916090674380800\pi s^3 \\
& - 11895806361600s^4+43698880512000\pi s^4-1439288524800s^5\big)
\end{aligned}
\tag{B.2}
$$

Appendix C

Padé approximants of higher order

In this appendix we present a complete list of the possible Padé approximants that can be calculated with equation B.1.

$$\Omega^{[1/1]}(\varepsilon,s)=\frac{4\varepsilon}{3\left(1+\frac{1}{48}(16-15\pi+16s)\varepsilon\right)} \tag{C.1}$$

$$\Omega^{[1/2]}(\varepsilon,s)=\frac{4\varepsilon}{3\left(1+\frac{1}{48}(16-15\pi+16s)\varepsilon+\frac{\left(-39296+3375\pi^2+3840s+2400\pi s+3840s^2\right)\varepsilon^2}{34560}\right)} \tag{C.2}$$

$$\Omega^{[1/3]}(\varepsilon,s) = 4\varepsilon\left(3\left(1+\frac{1}{48}(16-15\pi+16s)\varepsilon\right.\right.$$
$$+\ \frac{\left(-39296+3375\pi^2+3840s+2400\pi s+3840s^2\right)\varepsilon^2}{34560}$$
$$+\ \frac{\varepsilon^3}{1658880}\left(1497088+643920\pi-54000\pi^2-50625\pi^3+1746944s\right.$$
$$\left.\left.\left.+\ 76800\pi s-126000\pi^2 s-61440s^2+134400\pi s^2+61440s^3\right)\right)\right)^{-1} \tag{C.3}$$

$$\Omega^{[1/4]}(\varepsilon,s) = 4\varepsilon\left(3\left(1+\frac{1}{48}(16-15\pi+16s)\varepsilon\right.\right.$$
$$+\ \frac{\left(-39296+3375\pi^2+3840s+2400\pi s+3840s^2\right)\varepsilon^2}{34560}$$
$$+\ \frac{\varepsilon^3}{1658880}\left(1497088+643920\pi-54000\pi^2-50625\pi^3+1746944s\right.$$
$$\left.+\ 76800\pi s-126000\pi^2 s-61440s^2+134400\pi s^2+61440s^3\right)$$
$$+\ \frac{\varepsilon^4}{8360755200}\left(-3765628928-2318400000\pi-1009260000\pi^2\right.$$
$$+\ 170100000\pi^3+79734375\pi^4-8797716480s-3672614400\pi s$$
$$-\ 30240000\pi^2 s+340200000\pi^3 s+4035870720s^2+645120000\pi s^2$$
$$\left.\left.\left.-\ 262080000\pi^2 s^2-309657600s^3+387072000\pi s^3+103219200s^4\right)\right)\right)^{-1} \tag{C.4}$$

$$\Omega^{[2/1]}(\varepsilon,s) = \frac{\frac{4\varepsilon}{3}+\frac{\left(-39296+3375\pi^2+3840s+2400\pi s+3840s^2\right)\varepsilon^2}{540(-16+15\pi-16s)}}{1+\frac{2(-1348+225\pi-120s+300\pi s)\varepsilon}{45(-16+15\pi-16s)}} \tag{C.5}$$

$$\Omega^{[2/2]}(\varepsilon,s) = \tag{C.6}$$

$$
\begin{aligned}
\Omega^{[2/3]}(\varepsilon,s) = \Bigg(&\frac{4\varepsilon}{3} + \big(\big(-3765628928 - 2318400000\pi - 1009260000\pi^2 \\
&+ 170100000\pi^3 + 79734375\pi^4 - 8797716480s - 3672614400\pi s \\
&- 30240000\pi^2 s + 340200000\pi^3 s + 4035870720s^2 + 645120000\pi s^2 \\
&- 262080000\pi^2 s^2 - 309657600s^3 + 387072000\pi s^3 + 103219200s^4\big)\,\varepsilon^2\big)\,/ \\
&\big(3780\,\big(-1497088 - 643920\pi + 54000\pi^2 + 50625\pi^3 - 1746944s \\
&-76800\pi s + 126000\pi^2 s + 61440s^2 - 134400\pi s^2 - 61440s^3\big)\big)\Bigg)\,(1+ \\
&\big(\big(-392546048 - 65142000\pi + 5977125\pi^2 + 10631250\pi^3 - 890480640s \\
&- 133249200\pi s + 24570000\pi^2 s + 14175000\pi^3 s + 75264000s^2 \\
&+ 12096000\pi s^2 + 10080000\pi^2 s^2 - 19353600s^3 + 16128000\pi s^3\big)\,\varepsilon\big)\,/ \\
&\big(315\,\big(-1497088 - 643920\pi + 54000\pi^2 + 50625\pi^3 - 1746944s \\
&- 76800\pi s + 126000\pi^2 s + 61440s^2 - 134400\pi s^2 - 61440s^3\big)\big) \\
&+ \big(\big(21972307968 + 12282149760\pi - 1974672000\pi^2 - 704851875\pi^3 \\
&+ 14955008000s + 923462400\pi s - 2927484000\pi^2 s + 567000000\pi^3 s \\
&- 11268096000s^2 - 7545619200\pi s^2 - 584640000\pi^2 s^2 + 604800000\pi^3 s^2 \\
&+ 1950842880s^3 + 1032192000\pi s^3 - 645120000\pi^2 s^3 - 206438400s^4\big)\,\varepsilon^2\big)\,/ \\
&\big(15120\,\big(-1497088 - 643920\pi + 54000\pi^2 + 50625\pi^3 - 1746944s \\
&- 76800\pi s + 126000\pi^2 s + 61440s^2 - 134400\pi s^2 - 61440s^3\big)\big) \\
&+ \big(\big(-341247875072 - 434910873600\pi + 1528905375\pi^2 + 34020000000\pi^3 \\
&- 851421941760s - 523412121600\pi s + 433377000000\pi^2 s + 34955550000\pi^3 s \\
&- 1984223416320s^2 - 514027584000\pi s^2 + 147843360000\pi^2 s^2 + 27216000000\pi^3 s^2 \\
&- 11302502400s^3 - 288175104000\pi s^3 - 14515200000\pi^2 s^3 + 24192000000\pi^3 s^3 \\
&- 49545216000s^4 + 15482880000\pi s^4\big)\,\varepsilon^3\big)\,/\,\big(680400\,\big(-1497088 - 643920\pi \\
&+ 54000\pi^2 + 50625\pi^3 - 1746944s - 76800\pi s + 126000\pi^2 s + 61440s^2 \\
&- 134400\pi s^2 - 61440s^3\big)\big)\big)
\end{aligned}
$$

$$
\begin{aligned}
\Omega^{[3/1]}(\varepsilon,s) = \Bigg(&\frac{4\varepsilon}{3} + \big(\big(-132864 - 3405\pi + 6750\pi^2 - 144640s - 6000\pi s + 9000\pi^2 s \\
&+ 11520s^2 - 9600\pi s^2\big)\,\varepsilon^2\big) \,/\, (72(-1348 + 225\pi - 120s + 300\pi s)) \\
&+ \big(\big(-74054656 - 11394000\pi + 6712875\pi^2 + 67522560s - 1966800\pi s \\
&- 5400000\pi^2 s + 43223040s^2 + 2880000\pi s^2 - 5760000\pi^2 s^2 \\
&- 1843200s^3\big)\,\varepsilon^3\big) \,/\, (51840(-1348 + 225\pi - 120s + 300\pi s))\Big) \\
&/ \left(1 + \frac{\left(-176000 + 44235\pi - 191616s + 14400\pi s + 7680s^2\right)\varepsilon}{96(-1348 + 225\pi - 120s + 300\pi s)}\right)
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
\Omega^{[3/2]}(\varepsilon,s) = \Bigg(&\frac{4\varepsilon}{3} + \big(\big(21972307968 + 12282149760\pi - 1974672000\pi^2 - 704851875\pi^3 \\
&+ 14955008000s + 923462400\pi s - 2927484000\pi^2 s + 567000000\pi^3 s - 11268096000s^2 \\
&- 7545619200\pi s^2 - 584640000\pi^2 s^2 + 604800000\pi^3 s^2 + 1950842880s^3 \\
&+ 1032192000\pi s^3 - 645120000\pi^2 s^3 - 206438400s^4\big)\,\varepsilon^2\big) \,/\, \big(252\,\big(74054656 \\
&+ 11394000\pi - 6712875\pi^2 - 67522560s + 1966800\pi s + 5400000\pi^2 s - 43223040s^2 \\
&- 2880000\pi s^2 + 5760000\pi^2 s^2 + 1843200s^3\big)\big) + \big(\big(341247875072 + 434910873600\pi \\
&- 1528905375\pi^2 - 34020000000\pi^3 + 851421941760s + 523412121600\pi s \\
&- 433337700000\pi^2 s - 34955550000\pi^3 s + 1984223416320s^2 + 514027584000\pi s^2 \\
&- 147843360000\pi^2 s^2 - 27216000000\pi^3 s^2 + 11302502400s^3 + 288175104000\pi s^3 \\
&+ 14515200000\pi^2 s^3 - 24192000000\pi^3 s^3 + 49545216000s^4 - 15482880000\pi s^4\big)\,\varepsilon^3\big) \,/ \\
&\big(11340\,\big(74054656 + 11394000\pi - 6712875\pi^2 - 67522560s + 1966800\pi s + 5400000\pi^2 s \\
&- 43223040s^2 - 2880000\pi s^2 + 5760000\pi^2 s^2 + 1843200s^3\big)\big)\Bigg)\,\Big(1 + \big(2\,\big(945825920 \\
&+ 180704340\pi - 122590125\pi^2 + 490206336s + 297179400\pi s - 102532500\pi^2 s \\
&- 739737600s^2 - 97171200\pi s^2 + 30240000\pi^2 s^2 - 83865600s^3 + 16128000\pi s^3\big)\,\varepsilon\big) \,/ \\
&\big(21\,\big(74054656 + 11394000\pi - 6712875\pi^2 - 67522560s + 1966800\pi s + 5400000\pi^2 s \\
&- 43223040s^2 - 2880000\pi s^2 + 5760000\pi^2 s^2 + 1843200s^3\big)\big) + \big(\big(-40154343424 \\
&+ 7618126080\pi + 1392490575\pi^2 + 179372756992s + 38285172480\pi s \\
&- 21447216000\pi^2 s + 186239434752s^2 + 38280883200\pi s^2 - 17160192000\pi^2 s^2 \\
&- 23079813120s^3 + 7741440000\pi s^3 + 412876800s^4\big)\,\varepsilon^2\big) \,/\, \big(1008\,\big(74054656 \\
&+ 11394000\pi - 6712875\pi^2 - 67522560s + 1966800\pi s + 5400000\pi^2 s - 43223040s^2 \\
&- 2880000\pi s^2 + 5760000\pi^2 s^2 + 1843200s^3\big)\big)\Big)^{-1}
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
\Omega^{[4/1]}(\varepsilon,s) = \Bigg(& \frac{4\varepsilon}{3} + \big(\big(-75609344 + 3909360\pi + 4644675\pi^2 - 58935296s + 4332720\pi s \\
& + 1512000\pi^2 s + 10278912s^2 - 806400\pi s^2 - 860160s^3\big)\,\varepsilon^2\big)\,(252\,(-176000 \\
& + 44235\pi - 191616s + 14400\pi s + 7680s^2)\big)^{-1} + \big(\big(-945825920 - 180704340\pi \\
& + 122590125\pi^2 - 490206336s - 297179400\pi s + 102532500\pi^2 s + 739737600s^2 \\
& + 97171200\pi s^2 - 30240000\pi^2 s^2 + 83865600s^3 - 16128000\pi s^3\big)\,\varepsilon^3\big)\,/\,(5670 \\
& \big(-176000 + 44235\pi - 191616s + 14400\pi s + 7680s^2\big)\big) + \big(\big(-40154343424 \\
& + 7618126080\pi + 1392490575\pi^2 + 179372756992s + 38285172480\pi s \\
& - 21447216000\pi^2 s + 186239434752s^2 + 38280883200\pi s^2 - 17160192000\pi^2 s^2 \\
& - 23079813120s^3 + 7741440000\pi s^3 + 412876800s^4\big)\,\varepsilon^4\big)\,/\,(544320\,(-176000 \\
& + 44235\pi - 191616s + 14400\pi s + 7680s^2)\big)\Bigg)\cdot \\
& \left(1 + \frac{4\left(-1489396 + 427245\pi - 1564192s + 484680\pi s - 161280s^2\right)\varepsilon}{21\,(-176000 + 44235\pi - 191616s + 14400\pi s + 7680s^2)}\right)^{-1}
\end{aligned}
\tag{C.10}
$$

Appendix D

Motion of massive particles around a Kerr black hole

Looking back at section 2.1, for massive particles, the following conditions are satisfied:

$$\eta = mc$$
$$\lambda = \frac{\tau}{m}$$
$$\frac{dr}{d\lambda} = m\frac{dr}{d\tau}$$

Thus, we rewrite the equations of motion:

$$\Longrightarrow m^2\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E'}{c}\right)^2 - m^2c^2 + \frac{1}{r}\left(2\mu m^2c^2\right)$$
$$+\frac{1}{r^2}\left(\left(a\frac{E'}{c}\right)^2 - (h')^2 - a^2m^2c^2\right) + \frac{1}{r^3}2\mu\left(h' - a\frac{E'}{c}\right)^2$$

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{c}\right)^2 - c^2 + \frac{2\mu c^2}{r} + \frac{1}{r^2}\left(\left(a\frac{E}{c}\right)^2 - (h)^2 - a^2c^2\right) + \frac{2\mu}{r^3}\left(h - a\frac{E}{c}\right)^2 \tag{D.1}$$

From equation D.1, we can easily recognize the effective potential, which can be written as:

$$V_{\text{eff}}(r,h,E) = -\frac{\mu c^2}{r} + \frac{1}{2r^2}\left(h^2 + a^2\left(c^2 - \frac{E^2}{c^2}\right)\right) - \frac{\mu}{r^3}\left(h - a\frac{E}{c}\right)^2, \tag{D.2}$$

The equation containing the velocity of the particle can be rewritten using the effective potential. If we derive the following equation (D.3) with respect to the proper time, an equation for the particle's acceleration is obtained.

$$\Longrightarrow \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r,h,E) = \frac{1}{2}\left(\left(\frac{E}{c}\right)^2 - c^2\right) \tag{D.3}$$

$$\frac{d^2r}{d\tau^2} = -\frac{dV_{\text{eff}}}{dr}$$

Now, the equations D.1, 2.6 and 2.7 can be expressed in terms of the proper time and the coordinate time, using the following relations:

$$\begin{aligned}
&\tfrac{dr}{d\tau} = \tfrac{dr}{dt}\tfrac{dt}{d\tau} \longrightarrow \tfrac{dr}{dt} = \tfrac{dr}{d\tau}\tfrac{d\tau}{dt}\\
&\tfrac{dt}{d\lambda} = m\tfrac{dt}{d\tau}\\
&\tfrac{d\phi}{d\lambda} = m\tfrac{d\phi}{d\tau}\\
&\tfrac{d\phi}{d\tau} = \tfrac{d\phi}{dt}\tfrac{dt}{d\tau} \longrightarrow \tfrac{d\phi}{dt} = \tfrac{d\phi}{d\tau}\tfrac{d\tau}{dt}
\end{aligned}$$

$$\frac{dt}{d\tau} = \frac{1}{c\Delta}\left[\left(a^2 + \frac{2a^2\mu}{r} + r^2\right)\frac{E}{c} - \frac{2a\mu}{r}h\right] \tag{D.4}$$

$$\frac{dr}{dt} = \frac{-c\Delta\left[\left(\frac{E}{c}\right)^2 - c^2 + \frac{2\mu c^2}{r} + \frac{1}{r^2}\left(\left(a\frac{E}{c}\right)^2 - (h)^2 - a^2c^2\right) + \frac{2\mu}{r^3}\left(h - a\frac{E}{c}\right)^2\right]^{1/2}}{\left[\left(a^2 + \frac{2a^2\mu}{r} + r^2\right)\frac{E}{c} - \frac{2a\mu}{r}h\right]} \tag{D.5}$$

Note that we have chosen by hand the negative sign of the radical expression in D.5, assuming that the particle must approach the gravitational source.

$$\frac{d\phi}{d\tau} = \frac{1}{\Delta}\left[\left(\frac{2a\mu}{r}\right)\frac{E}{c} - \left(1 - \frac{2\mu}{r}\right)h\right]$$

$$\frac{d\phi}{dt} = c\left[\left(\frac{2a\mu}{r}\right)\frac{E}{c} - \left(1 - \frac{2\mu}{r}\right)h\right] \times \left[\left(a^2 + \frac{2a^2\mu}{r} + r^2\right)\frac{E}{c} - \frac{2a\mu}{r}h\right]^{-1} \tag{D.6}$$

Now that we have the general equations of motion for a massive particle (remember that E has dimensions of energy per unit mass and h, angular momentum per unit mass); we will study a particular case, a particle at rest at infinity (very far away from the gravitational source) with no angular momentum falls towards the black hole.

$$h = 0$$
$$E = c^2 \quad \text{(Energy at rest)}$$

The previous initial conditions lead to the following differential equations of motion:

$$\frac{dr}{dt} = \frac{-\Delta\left[\frac{2\mu c^2}{r}\left(1 + \frac{a^2}{c^2}\right)\right]^{1/2}}{\left(a^2 + \frac{2a^2\mu}{r} + r^2\right)} \tag{D.7}$$

$$\frac{d\phi}{dt} = \frac{\left(\frac{2a\mu c}{r}\right)}{\left(a^2 + \frac{2a^2\mu}{r} + r^2\right)} \tag{D.8}$$

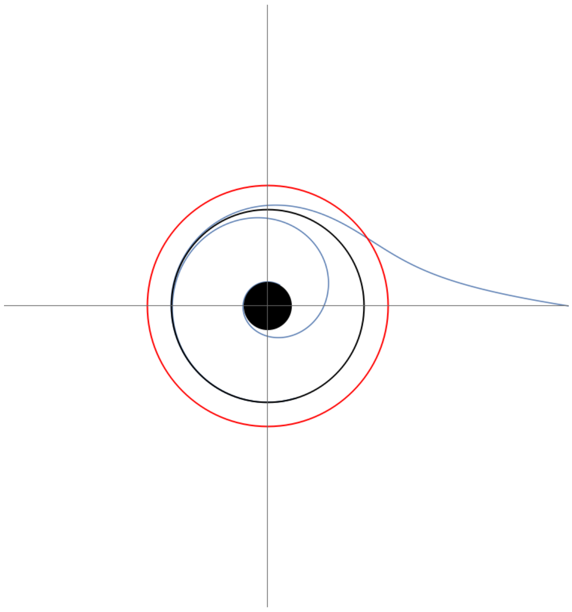

Figure 4.1: Massive particle falling inside a Kerr black hole.

Finally, for a rigorous analysis of a particles motion around a Kerr black hole, it is necessary to study the infinite redshift surfaces and the event horizons. First, the infinite redshift surfaces are given by the relation $g_{\mathrm{tt}} = 0$:

$$\implies 1 - \frac{2\mu r}{\rho^2} = 0$$

$$\rho^2 = r^2 + a^2 \cos^2\theta$$

$$r^2 + a^2 \cos^2\theta - 2\mu r = 0$$

$$r_s^{\pm} = \frac{2\mu \pm \sqrt{4\mu^2 - 4a^2\cos^2\theta}}{2}$$

$$r_s^{\pm} = \mu \pm \sqrt{\mu^2 - a^2\cos^2\theta} \tag{D.9}$$

For $\theta = \frac{\pi}{2}$, we have a single infinite redshift surface:

$$\Longrightarrow r_s = 2\mu \tag{D.10}$$

Now, the condition for event horizons is $g^{\mathrm{rr}} = 0$.

$$\Longrightarrow -\frac{\Delta}{\rho^2} = 0$$

$$\Delta = r^2 - 2\mu r + a^2 = 0$$

$$r_{\pm} = \frac{2\mu \pm \sqrt{4\mu^2 - 4a^2}}{2}$$

$$r_{\pm} = \mu \pm \sqrt{\mu^2 - a^2} \tag{D.11}$$

It is important to note that the value of a must not exceed that of μ ($\mu^2 \geqslant a^2$); for an "extreme" Kerr black hole $\mu^2 = a^2$, thus a single event horizon with radius $r = \mu$ exists.

Note that in the case where $a = 0$, we have a single infinite redshift surface, which is the same as the event horizon:

$$r_s^{\pm} = r_{\pm} = 2\mu$$

This reproduces the results for the Schwarzschild metric!

4.1 Circular orbits in the Kerr metric

4.1.1 Stable circular orbits

In this section we will study particles orbiting a Kerr black hole in the equatorial plane, for circular orbits the following conditions must be met:

$$\theta = \frac{\pi}{2}$$

$$p^{\theta} = 0$$

$$\dot{r} = \frac{dr}{d\tau} = 0, \; \ddot{r} = 0$$

Let's remember equations D.1 and D.2, which should fulfill the conditions above. Since $\frac{dr}{d\tau} = 0$, then the expression for the effective potential reduces to:

$$\Longrightarrow V_{\text{eff}}(r, h, E) = \frac{1}{2}\left(\left(\frac{E}{c}\right)^2 - c^2\right),$$

$$\left[\frac{dV_{\text{eff}}}{dr}\right]_{r=r_c} = 0, \tag{D.12}$$

where r_c is the critical radius, or the radius of the smallest stable circular orbit.

Now, let's try a very well known change of variable, which will make the rest of the calculation easier:

$$u(r) = \frac{1}{r}$$

$$\frac{dV_{\text{eff}}}{dr} = \frac{dV_{\text{eff}}}{du}\frac{du}{dr} = -\frac{1}{r^2}\frac{dV_{\text{eff}}}{du} = -u^2\frac{dV_{\text{eff}}}{du} = 0$$

$$\implies \left[\frac{dV_{\text{eff}}}{du}\right]_{u=u_c=\frac{1}{r_c}} = 0$$

We are going to find a simple equation for the radius of the smallest stable circular orbit around a Kerr black hole, given its spin. First, we rewrite equation D.2 in terms of $u(r)$ and taking the derivative, we get the following equations:

$$\begin{aligned} V_{\text{eff}}(u,h,E) = -\mu c^2 u + \frac{1}{2}u^2\left(h^2 + a^2\left(c^2 - \frac{E^2}{c^2}\right)\right) - \mu u^3\left(h - a\frac{E}{c}\right)^2 \\ = \frac{1}{2}\left(\left(\frac{E}{c}\right)^2 - c^2\right) \end{aligned} \tag{D.13}$$

$$\frac{dV_{\text{eff}}}{du} = -\mu c^2 + u\left(h^2 + a^2\left(c^2 - \frac{E^2}{c^2}\right)\right) - 3\mu u^2\left(h - a\frac{E}{c}\right)^2$$

Let's consider the relation $x \equiv h - a\frac{E}{c}$, which we will replace in the equations for the effective potential; we need to rewrite them:

$$\Longrightarrow V_{\text{eff}}(u,h,E) = -\mu c^2 u + \frac{1}{2}u^2\left(x^2 + \frac{2xaE}{c} + a^2c^2\right) - \mu u^3 x^2 \\ = \frac{1}{2}\left(\left(\frac{E}{c}\right)^2 - c^2\right), \tag{D.14}$$

then

$$\frac{dV_{\text{eff}}}{du} = -\mu c^2 + u\left(\left(h - a\frac{E}{c}\right)\left(h + a\frac{E}{c}\right) + a^2c^2\right) - 3\mu u^2\left(h - a\frac{E}{c}\right)^2,$$

$$\Longrightarrow \left[\frac{dV_{\text{eff}}}{du}\right]_{u=u_c} = \left[-\mu c^2 + u\left(x^2 + \frac{2xaE}{c} + a^2c^2\right) - 3\mu u^2 x^2\right]_{u=u_c} = 0 \tag{D.15}$$

From now on, we will assume $u = u_c$ for simplicity, since all the equations are intended to find the critical radius. First step is to multiply equation (D.15)$\times u$:

$$-\mu c^2 u + u^2\left(x^2 + \frac{2xaE}{c} + a^2c^2\right) - 3\mu u^3 x^2 = 0 \tag{D.16}$$

Multiply equation (D.14)$\times 2$:

$$-2\mu c^2 u + u^2\left(x^2 + \frac{2xaE}{c} + a^2c^2\right) - 2\mu u^3 x^2 = \left(\frac{E}{c}\right)^2 - c^2 \tag{D.17}$$

Substract D.16 from D.17 to get an expression for the energy:

$$-\mu c^2 u + \mu u^3 x^2 = \left(\frac{E}{c}\right)^2 - c^2$$

$$\left(\frac{E}{c}\right)^2 = c^2(1 - \mu u) + \mu u^3 x^2 \tag{D.18}$$

From equation D.15,

$$\frac{2xaE}{c}u = x^2u(3\mu u - 1) - c^2\left(a^2u - \mu\right)$$

and squaring it:

$$4x^2a^2\left(\frac{E}{c}\right)^2u^2 = x^4u^2(3\mu u - 1)^2 + c^4\left(a^2u - \mu\right)^2 - 2x^2c^2u(3\mu u - 1)\left(a^2u - \mu\right) \qquad \text{(D.19)}$$

Now, we replace D.18 into D.19:

$$4x^2a^2\left(c^2(1 - \mu u) + \mu u^3x^2\right)u^2 = x^4u^2(3\mu u - 1)^2 + c^4\left(a^2u - \mu\right)^2 - 2x^2c^2u(3\mu u - 1)\left(a^2u - \mu\right)$$

and rearranging to get a quadratic equation which we can easily solve:

$$\begin{aligned}\left((3\mu u - 1)^2 - 4\mu u^3a^2\right)u^2x^4 - 2uc^2\left[(3\mu u - 1)\left(a^2u - \mu\right) + 2a^2u(1 - \mu u)\right]x^2 \\ + c^4\left(a^2u - \mu\right)^2 = 0\end{aligned} \qquad \text{(D.20)}$$

$$\begin{aligned} x^2 = \{2uc^2\left[(3\mu u - 1)\left(a^2u - \mu\right) + 2a^2u(1 - \mu u)\right] \\ \pm[4u^2c^4\left((3\mu u - 1)\left(a^2u - \mu\right) + 2a^2u(1 - \mu u)\right)^2 \\ -4u^2c^4\left((3\mu u - 1)^2 - 4\mu u^3a^2\right)\left(ua^2 - \mu\right)^2]^{1/2}\} \\ \times\left\{2u^2\left[(3\mu u - 1)^2 - 4\mu u^3a^2\right]\right\}^{-1}\end{aligned}$$

$$x^2 = \frac{c^2}{u}\{\left(\mu u^2 a^2 + a^2 u + \mu - 3\mu^2 u\right)$$
$$\pm\left[-16\mu^2 u^4 a^4 + 8\mu u^3 a^4 - 16u^2\mu^2 a^2 + 16\mu^3 u^3 a^2 + 4ua^2\mu + 4\mu u^5 a^6\right]^{1/2}\}$$
$$\times\left[(3\mu u - 1)^2 - 4\mu u^3 a^2\right]^{-1}$$

A little bit of algebra is needed to get the equation above, now in order to simplify our result we will introduce yet another relation [6]: $\delta = a^2u^2 - 2\mu u + 1$, and squaring it gives an expression very similar to the one inside the square root:

$$\delta^2 = a^4u^4 + 4\mu^2u^2 + 1 - 4a^2\mu u^3 - 4\mu u + 2a^2u^2$$

$$4a^2\mu\delta^2 u = -16\mu^2u^4a^4 + 8\mu u^3a^4 - 16u^2\mu^2a^2 + 16\mu^3u^3a^2 + 4ua^2\mu + 4\mu u^5a^6$$

Then, x^2 can be written in a much simpler form:

$$\Longrightarrow x^2 = \frac{\frac{c^2}{u}\left(\mu u^2 a^2 + a^2 u + \mu - 3\mu^2 u \pm 2a\delta\sqrt{\mu u}\right)}{(3\mu u - 1)^2 - 4\mu u^3 a^2} \tag{D.21}$$

Now let's consider the following relation that will help us study both cases (choosing the $+$ and $-$ signs):

$$R_\pm \equiv 1 - 3\mu u \pm 2a\sqrt{\mu u^3}$$

$$R_+R_- = (3\mu u - 1)^2 - 4\mu u^3 a^2$$

$$R_\pm\delta - R_+R_- = a^2u^2 + \mu u - 3\mu^2u^2 + a^2\mu u^3 \pm 2au\sqrt{\mu u}\delta$$

$$R_\pm\delta - R_+R_- = u\left(a^2u + \mu - 3\mu^2 u + a^2\mu u^2 \pm 2a\sqrt{\mu u}\delta\right)$$

$$\implies x^2 = \left(\frac{c^2}{u^2}\right)\frac{R_\pm\delta - R_+R_-}{R_+R_-} \tag{D.22}$$

With R_+ in D.22:

$$x^2 = \left(\frac{c^2}{u^2}\right)\frac{\delta - R_-}{R_-}$$

With R_- in D.22:

$$x^2 = \left(\frac{c^2}{u^2}\right)\frac{\delta - R_+}{R_+}$$

Therefore, we can rewrite D.22 in a much simpler form:

$$\implies x^2 = \left(\frac{c^2}{u^2}\right)\frac{(\delta - R_\mp)}{R_\mp} \tag{D.23}$$

Let's calculate what $\delta - R_\mp$ is:

$$\delta - R_\mp = a^2u^2 - 2\mu u + 1 - \left(1 - 3\mu u \mp 2a\sqrt{\mu u^3}\right)$$

$$\delta - R_{\mp} = u\left(a\sqrt{u} \pm \sqrt{\mu}\right)^2$$

$$\Longrightarrow x = -\frac{c}{\sqrt{u}} \frac{a\sqrt{u} \pm \sqrt{\mu}}{\left(1 - 3\mu u \mp 2a\sqrt{\mu u^3}\right)^{1/2}} \tag{D.24}$$

Note: remember that $x = h - a\frac{E}{c}$, so we choose the '$-$' sign in D.24, because otherwise an unstable orbit would occur, clearly $x < 0$.

Now, replacing our result of x^2 in equation D.18, gives us an expression for the energy, which entirely depends on u, a and μ:

$$E = \frac{c^2\left(1 - 2\mu u \mp au\sqrt{\mu u}\right)}{\left(1 - 3\mu u \mp 2au\sqrt{\mu u}\right)^{1/2}} \tag{D.25}$$

If we choose the '$+$' sign, D.25 gives the energy per unit mass of the particle orbiting in the same direction as the gravitational source, otherwise its the energy of the retrograde orbit.

The same can be done for the angular momentum:

$$h = x + a\frac{E}{c}$$

$$h = \frac{\mp c\sqrt{\mu}\left(1 \pm 2au\sqrt{\mu u} + a^2u^2\right)}{\sqrt{u}\left(1 - 3\mu u \mp 2au\sqrt{\mu u}\right)^{1/2}} \tag{D.26}$$

Now, we can finally attempt to calculate a critical radius, we know that for a stable orbit, the second derivative of the potential must be positive, let's see what condition must be met for $\frac{d^2V_{\text{eff}}}{du^2}$.

$$\frac{dV_{\text{eff}}}{dr} = \frac{dV_{\text{eff}}}{du}\frac{du}{dr} = -\frac{1}{r^2}\frac{dV_{\text{eff}}}{du}$$

$$\frac{d^2V_{\text{eff}}}{dr^2} = \frac{2}{r^3}\frac{dV_{\text{eff}}}{du} + \frac{1}{r^4}\frac{d^2V_{\text{eff}}}{du^2}$$

$$\left[\frac{d^2V_{\text{eff}}}{dr^2}\right]_{r=r_c} \geqslant 0 \Rightarrow \left[\frac{d^2V_{\text{eff}}}{du^2}\right]_{u=u_c} \geqslant 0$$

Knowing that $\frac{d^2V_{\text{eff}}}{du^2} \geqslant 0$, we can proceed to calculate its expression using D.25 and D.26.

$$\frac{d^2V_{\text{eff}}}{du^2} = \left(x\left(h + a\frac{E}{c}\right) + a^2c^2\right) - 6\mu u x^2$$

$$h + a\frac{E}{c} = \frac{c\left(\mp\sqrt{\mu} - 4a\mu u^{3/2} \mp 2a^2u^2\sqrt{\mu} + a\sqrt{u}\right)}{\sqrt{u}\left(1 - 3\mu u \mp 2au\sqrt{\mu u}\right)^{1/2}}$$

$$x\left(h + a\frac{E}{c}\right) = -\frac{c^2\left(a\sqrt{u} \pm \sqrt{\mu}\right)\left(\mp\sqrt{\mu} - 4a\mu u^{3/2} \mp 2a^2u^2\sqrt{\mu} + a\sqrt{u}\right)}{u\left|1 - 3\mu u \mp 2au\sqrt{\mu u}\right|}$$

Note that we previously chose the sign for x, so the denominator has to be positive.

$$\frac{d^2V_{\text{eff}}}{du^2} = \left(a^2c^2 - \frac{c^2\left(a\sqrt{u} \pm \sqrt{\mu}\right)\left(\mp\sqrt{\mu} - 4a\mu u^{3/2} \mp 2a^2u^2\sqrt{\mu} + a\sqrt{u}\right)}{u\left|1 - 3\mu u \mp 2au\sqrt{\mu u}\right|}\right)$$
$$-6\mu c^2\frac{\left(a\sqrt{u} \pm \sqrt{\mu}\right)^2}{\left|1 - 3\mu u \mp 2a\sqrt{\mu u^3}\right|}$$

$$\left[\frac{d^2V_{\text{eff}}}{du^2}\right]_{u=u_c} = \left[\frac{c^2\mu\left(1 \mp 8au^{3/2}\sqrt{\mu} - 6\mu u - 3a^2u^2\right)}{u\left|1 - 3\mu u \mp 2au\sqrt{\mu u}\right|}\right]_{u=u_c} \geqslant 0$$

Since the expression has to be greater or equal to zero, then numerator has to be greater or equal to zero as well:

$$\Longrightarrow 1 \mp 8au_c^{3/2}\sqrt{\mu} - 6\mu u_c - 3a^2u_c^2 \geqslant 0$$

Now, we can go back to working with $r = r_c$:

$$1 \mp 8ar_c^{-3/2}\sqrt{\mu} - 6\mu r_c^{-1} - 3a^2r_c^{-2} \geqslant 0,$$

multiplying by r_c^2, and considering $a = \mu s$:

$$r_c^2 - 6\mu r_c - 3a^2 \mp 8a\sqrt{r_c\mu} \geqslant 0$$

$$r_c^2 - 6\mu r_c - 3\mu^2s^2 \mp 8\mu s\sqrt{r_c\mu} \geqslant 0$$

$$\left(\frac{r_c}{\mu}\right)^2 - 6\left(\frac{r_c}{\mu}\right) - 3s^2 \mp 8s\sqrt{\frac{r_c}{\mu}} \geqslant 0$$

$$r_0 = \frac{r_c}{\mu}$$

$$\Longrightarrow r_0^2 - 6r_0 - 3s^2 \mp 8s\sqrt{r_0} \geqslant 0 \tag{D.27}$$

Finally, we arrive to an inequality which determines the smallest possible circular orbit of a particle orbiting a rotating black hole, when equalling to zero as in D.28, we get precisely the radius of the smallest orbit:

$$r_0^2 - 6r_0 - 3s^2 \mp 8s\sqrt{r_0} = 0 \tag{D.28}$$

To solve this equation we will use *Mathematica*, which gives us a not very friendly but analytic solution. Four roots result from the computation, but only two of them produce real values, these two are shown below. Here, r_0^+ refers to the prograde motion and r_0^- to the retrograde motion.

$$
\begin{aligned}
r_0^+ = 3 + & \left(3 + s^2 + \frac{9 - 10s^2 + s^4}{\left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}}\right. \\
& \left. + \left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}\right)^{1/2} \\
-\frac{1}{2} & \left(72 + 8\left(-6+s^2\right) - \frac{4\left(9 - 10s^2 + s^4\right)}{\left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}}\right. \\
& -4\left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3} \\
& + \left(64s^2\right) \times \left(3 + s^2 + \frac{9 - 10s^2 + s^4}{\left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}}\right. \\
& \left.\left. + \left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}\right)^{-1/2}\right)^{1/2}
\end{aligned}
\tag{D.29}
$$

$$r_0^- = 3 + \left(3 + s^2 + \frac{9 - 10s^2 + s^4}{\left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}} + \left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}\right)^{1/2}$$
$$+\frac{1}{2}\left(72 + 8\left(-6+s^2\right) - \frac{4\left(9 - 10s^2 + s^4\right)}{\left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}} - 4\left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3} + \left(64s^2\right) \times \left(3 + s^2 + \frac{9 - 10s^2 + s^4}{\left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}} + \left(27 - 45s^2 + 17s^4 + s^6 + 8\sqrt{s^6\left(-1+s^2\right)^2}\right)^{1/3}\right)^{-1/2}\right)^{1/2} \tag{D.30}$$

Now, let's study some particular cases, for an extreme Kerr black hole ($a = \mu \longrightarrow s = 1$) the equation reduces to:

$$r_0^2 - 6r_0 - 3 \mp 8\sqrt{r_0} = 0,$$

which has two possible solutions:

$$r_0^- = 9 \quad \longrightarrow \quad r_c^- = 9\mu$$
$$r_0^+ = 1 \quad \longrightarrow \quad r_c^+ = \mu$$

Using equations D.29 and D.30 we can calculate the smallest circular stable orbit given any spin other than $s = 1$, since it produces an indeterminate solution, below are listed some of the results for the critical radius:

s	r_0^-	r_0^+
0	6μ	6μ
$1/4$	6.79485μ	5.15554μ
$1/3$	7.05149μ	4.85883μ
$1/2$	7.55458μ	4.233μ
$3/5$	7.85069μ	3.82907μ
$4/5$	8.43179μ	2.90664μ
1	9μ	μ

Table 4.1: Critical radius. r_0^- retrograde orbit. r_0^+ prograde orbit.

Let's take a look at the case where there is no rotation of the gravitational source (with the condition $a = 0 \longrightarrow s = 0$), we get a the following solution:

$$r_0^2 - 6r_0 = 0$$

$$r_0 = 6 \quad \longrightarrow \quad r_c = 6\mu$$

$$\mu = \frac{GM}{c^2} \Longrightarrow r_s = \frac{2GM}{c^2} = 2\mu \Longrightarrow r_c = 3r_s$$

where r_s is the Schwarzschild radius, thus we prove that the theory is consistent. When there is no rotation in the metric, it reduces to the value for the smallest stable orbit in the Schwarzschild metric.

4.1.2 Released energy

In this section we will study the energy of a particle as it approaches its smallest possible stable circular orbit around a rotating black hole. The expression for the energy was already derived previously (equation D.25), first we will consider the case of an extreme Kerr black hole for a direct orbit.

$$E = \left[\frac{c^2 \left(1 - 2\mu u \mp a u \sqrt{\mu u}\right)}{\left(1 - 3\mu u \mp 2 a u \sqrt{\mu u}\right)^{1/2}} \right]_{u=u_c}$$

$$a = \mu \longrightarrow r_c^+ = \mu \Rightarrow u_c = \frac{1}{\mu}$$

$$E = \left[\frac{c^2 \left(1 - 2\mu u + \mu u \sqrt{\mu u}\right)}{\left(1 - 3\mu u + 2\mu u \sqrt{\mu u}\right)^{1/2}} \right]_{u=u_c}$$

Evidently, it is not possible to directly calculate the energy, since replacing $u_c = \frac{1}{\mu}$ directly into the energy's equation produces an indeterminacy. Thus, we need to apply L'Hôpital's rule to get a result, in this case the rule has to be applied twice, so we need to square our expression for the energy.

$$\frac{E^2}{c^4} = \lim_{u \to \frac{1}{\mu}} \frac{\left(1 - 2\mu u + \mu u \sqrt{\mu u}\right)^2}{\left(1 - 3\mu u + 2\mu u \sqrt{\mu u}\right)}$$

$$\frac{E^2}{c^4} = \lim_{u \to \frac{1}{\mu}} \left\{ \frac{\frac{d}{du}\left(1 - 2\mu u + \mu u \sqrt{\mu u}\right)^2}{\frac{d}{du}\left(1 - 3\mu u + 2\mu u \sqrt{\mu u}\right)} \right\}$$

$$\frac{E^2}{c^4} = 2 \lim_{u \to \frac{1}{\mu}} \left\{ \frac{\left(1 - 2\mu u + \mu u \sqrt{\mu u}\right)\left(-2\mu + \frac{3}{2}\mu^{3/2}\sqrt{u}\right)}{\left(-3\mu + 3\mu^{3/2}\sqrt{u}\right)} \right\}$$

$$\frac{E^2}{c^4} = 2\lim_{u\to\frac{1}{\mu}}\left\{\frac{\left(-2\mu+\frac{3}{2}\mu^{3/2}\sqrt{u}+4\mu^2u-5\mu^{5/2}u^{3/2}+\frac{3}{2}\mu^3u^2\right)}{\left(-3\mu+3\mu^{3/2}\sqrt{u}\right)}\right\}$$

$$\frac{E^2}{c^4} = \frac{2}{3}\lim_{u\to\frac{1}{\mu}}\left\{\frac{\frac{d}{du}\left(-2\mu+\frac{3}{2}\mu^{3/2}\sqrt{u}+4\mu^2u-5\mu^{5/2}u^{3/2}+\frac{3}{2}\mu^3u^2\right)}{\frac{d}{du}\left(-\mu+\mu^{3/2}\sqrt{u}\right)}\right\}$$

$$\frac{E^2}{c^4} = \frac{2}{3}\lim_{u\to\frac{1}{\mu}}\left\{\frac{\frac{3}{4}\mu^{3/2}u^{-1/2}+4\mu^2-\frac{15}{2}\mu^{5/2}u^{1/2}+3\mu^3u}{\frac{1}{2}\mu^{3/2}u^{-1/2}}\right\}$$

$$\frac{E^2}{c^4} = \frac{2}{3}\left(\frac{\frac{3}{4}\mu^2+4\mu^2-\frac{15}{2}\mu^2+3\mu^2}{\frac{1}{2}\mu^2}\right) = \frac{2}{3}\left(\frac{3}{2}+14-15\right) = \frac{1}{3}$$

$$E = \frac{1}{\sqrt{3}}c^2$$

$$E' = mE$$

Then,

$$\Longrightarrow E' = \frac{1}{\sqrt{3}}mc^2 \tag{D.31}$$

Now, an interesting value can be studied, the energy released by the particle when it falls from infinity (starting at rest with no angular momentum) towards the circular orbit $r_c^+ = \mu$:

$$\Delta \mathrm{E} = mc^2 - E' = mc^2\left(1-\frac{1}{\sqrt{3}}\right)$$

This means that the particle loses 42.265% of its rest energy!

The same can be calculated for the retrograde orbit:

$$a = \mu \longrightarrow r_c^- = 9\mu \Rightarrow u_c = \frac{1}{9\mu}$$

$$E = \left[\frac{c^2\left(1 - 2\mu u - \mu u\sqrt{\mu u}\right)}{\left(1 - 3\mu u - 2\mu u\sqrt{\mu u}\right)^{1/2}}\right]_{u=u_c}$$

$$\frac{E}{c^2} = \lim_{u \to \frac{1}{9\mu}} \left[\frac{\left(1 - 2\mu u - \mu^{3/2}u^{3/2}\right)}{\left(1 - 3\mu u - 2\mu^{3/2}u^{3/2}\right)^{1/2}}\right]$$

$$\frac{E}{c^2} = \frac{1 - \frac{2}{9} - \frac{1}{27}}{\left(1 - \frac{3}{9} - \frac{2}{27}\right)^{1/2}} = \frac{5\sqrt{3}}{9}$$

$$\Longrightarrow E' = \frac{5\sqrt{3}}{9}mc^2 \tag{D.32}$$

$$\Delta \text{E} = mc^2 - E' = mc^2\left(1 - \frac{5\sqrt{3}}{9}\right)$$

In this case the particle loses 3.7745% of its rest energy. In the case of the Schwarzschild metric, the energy released is 5.72% of the rest energy.

Following the same procedure we compute the energy release for various cases.

s	E_+	E_-
$1/4$	6.68927%	5.03230%
$1/3$	7.11336%	4.84448%
$1/2$	8.21180%	4.51422%
$3/5$	9.12133%	4.34010%
$4/5$	12.21387%	4.03470%
1	42.26500%	3.77500%

Table 4.2: Percentage of energy lost. E_+ prograde motion. E_- retrograde motion.

Appendix E

Stability of circular orbits of photons around a Kerr black hole

The stability condition is given by the nature of the second derivative of the effective potential, which we can derive from equation 2.15:

$$\frac{1}{h^2}\left(\frac{dr}{d\lambda}\right)^2 + V'_{eff}(r,b) = \frac{1}{b^2}, \tag{E.1}$$

where $V'_{eff}(r,b) = \frac{1}{r^2}\left[1-\left(\frac{a}{b}\right)^2 - \frac{2\mu}{r}\left(1-\frac{a}{b}\right)^2\right]$ is the effective potential for null geodesics.

For circular orbits, we have $\left(\frac{dr}{d\lambda}\right) = \left(\frac{d^2r}{d\lambda^2}\right) = 0$, then, the equation of the orbit would take the form $V'_{eff}(r,b) = \frac{1}{b^2}$. The second derivative of the effective potential with respect to r is:

$$\frac{d^2V'_{eff}(r,b)}{dr^2} = \frac{6}{r^4}\left(1-\left(\frac{a}{b}\right)^2\right) - \frac{24\mu}{r^5}\left(1-\frac{a}{b}\right)^2$$

$$\frac{d^2V'_{eff}(r,b)}{dr^2} = \frac{6}{r^4}\left(1-\left(\frac{a}{b}\right)^2\right)\left(1-\frac{4\mu}{r}\frac{\left(1-\frac{a}{b}\right)}{\left(1+\frac{a}{b}\right)}\right) \tag{E.2}$$

From this last equation, we know that $\left|\frac{a}{b}\right| < 1$, thus $1-\frac{a}{b} > 0$ for all possible values of a and b. Now, to analyze the behavior of $\frac{4\mu}{r}$, at the critical radius $r_c = 3\mu$:

$$\Rightarrow \frac{4\mu}{r_c} = \frac{4\mu}{3\mu} > 1$$

Therefore, the second derivative of the effective potential evaluated at r_c is negative, this indicates that no stable circular orbit occurs with $r_c = 3\mu$.

$$\left.\frac{d^2V'_{eff}(r,b)}{dr^2}\right|_{r=r_c} < 0 \tag{E.3}$$

Finally, for a circular orbit we consider:

$$\left.\frac{dV'_{eff}(r,b)}{dr}\right|_{r=r_c} = 0$$

This leads to the following expression for the critical radius [6]:

$$r_{c^{+}_{-}} = 2\mu\left[1+\cos\left(\frac{2}{3}\cos^{-1}\left(\pm\frac{a}{\mu}\right)\right)\right], \tag{E.4}$$

where the '$+$' sign represents retrograde orbits and '$-$' is for direct orbits.