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Open and Closed Strings in Superstring theories
II: The Strings Tensionless Limit

Tesis en torno a una hipótesis o problema de investigación

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**Open and Closed Strings in Superstring theories
II: The Strings Tensionless Limit**

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Dedicatoria

A mi familia, por su amor incondicional, apoyo constante y fe en mis sueños. Y a la memoria de mi padre, cuya inspiración y legado me han acompañado en este camino.

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Resumen

Esta tesis explora la teoría de las cuerdas en el límite de tensión cero. Se estudiarán las simetrías de la teoría y su dinámica. Se considerará a la teoría como un límite de la teoría de cuerdas, aunque también se estudiarán las cuerdas con tensión cero entendiéndolas como objetos fundamentales. Se encontró que las ecuaciones de movimiento para las cuerdas bosónicas y supercuerdas sin tensión se alinean con el Álgebra Conformal Galileana (GCA) y su contraparte supersimétrica (SGCA).

La cuantización canónica reveló características únicas: las cuerdas bosónicas sin tensión exhiben un vacío de estado de frontera de Neumann, mostrando que las cuerdas abiertas emergen de cuerdas cerradas, con todos los estados sin masa. Para las supercuerdas sin tensión, las transformaciones de Bogoliubov conectan los vacíos con y sin tensión, integrando modos fermiónicos y bosónicos en el vacío. Además, se propone un método basado en el álgebra $su(1,1)$ para calcular estos vacíos.

Palabras clave: cuerdas sin tensión, vacío, modos fermiónicos y bosónicos

Abstract

This thesis explores the theory of tensionless strings, examining their symmetries and dynamics as both a limit of tensile string theory and fundamental objects. It was found that the equations of motion for tensionless bosonic and superstrings align with the Galilean Conformal Algebra (GCA) and its supersymmetric counterpart (SGCA). Canonical quantization revealed unique features: tensionless bosonic strings exhibit a Neumann boundary state vacuum, showing that open strings emerge from closed strings, with all states being massless. For tensionless superstrings, the Bogoliubov transformation connect tensile and tensionless vacua, with the vacuum incorporating fermionic and bosonic modes. Additionally, a method based on the $su(1,1)$ algebra is proposed for calculating these vacua.

Key words: tensionless strings, vacuum, fermionic and boson modes.

Contents

1	The Bosonic String	14
1.1	Classical Description	14
1.1.1	The Nambu-Goto Action	14
1.1.2	Equations of Motion and Boundary Conditions	16
1.1.3	Polyakov Action	18
1.1.4	Equations of Motion and Mode Expansion	19
1.2	Quantum Description	22
1.2.1	Light Cone Quantization	22
1.2.2	Covariant Quantization	24
1.2.3	Boundary States	26
2	The Superstring	28
2.1	Classical Description	28
2.1.1	Boundary Conditions and Mode Expansions	29
2.1.2	Symmetries and Constrain Equations	31
2.2	Quantum Description	34
2.2.1	Covariant Quantization	35
3	The Tensionless Bosonic String	38
3.1	Classical Description	38
3.1.1	Tensionless Action	38
3.1.2	Equations of Motion and Mode Expansions	40
3.1.3	Generators	42
3.1.4	Limit from the Tensile String	43
3.2	Quantum Description	44
3.2.1	2d Galilean Conformal Algebra	44
3.2.2	Quantum Constrains	45
3.2.3	Induced Representation	46
3.2.4	Mass Spectrum	46
3.2.5	Induced Vacuum	47
4	The Tensionless Superstring	52
4.1	Classical Description	52
4.1.1	Tensionless Superstring Action	52
4.1.2	Symmetries	53
4.1.3	Equations of Motion and Mode Expansions	53
4.1.4	Super Galilean Conformal Algebra	54
4.1.5	Limit from the Tensile Superstring	55
4.2	Quantum Description	56

4.2.1	Quantum Constrains	56
4.2.2	Bogoliubov Transformation on the Worldsheet	56
4.2.3	Fermionic Squeezed Vacuum	57
5	Conclusions and Future Directions	60
A	$su(1,1)$ Algebra	62
B	Conformally flat 2D metric	63

List of Figures

1.1	From worldlines to worldsheets	15
1.2	String parameterization: the gauge condition $n \cdot X = \lambda\tau$ fixes the string coordinates to be the curve at the intersection of the hyperplane.	17
1.3	Closed/open string duality	27

Introduction

String theory has long been regarded as one of the most promising frameworks for unifying quantum mechanics and general relativity. At its core, string theory replaces point particles with one-dimensional extended objects known as strings. The tension of these strings, a fundamental parameter inversely proportional to the square of the characteristic length, determines their dynamical behavior. In the limit of infinite tension, string theory simplifies to the point-particle description. On the other side, in the tensionless limit, string theory enters into a regime that probes its ultra-quantum nature, offering insights into phenomena at extremely high energies.

The tensionless limit of string theory has been studied extensively since its inception in the 1970s, with foundational work by Schild [19] and subsequent refinements by Gross and Mende in the context of string scattering amplitudes [12]. This limit reveals an emergent symmetry structure far richer than that of the tensile theory. For bosonic strings, the familiar Virasoro symmetry on the worldsheet transitions to the Bondi-Metzner-Sachs algebra, or equivalently, its 2D Galilean Conformal Algebra counterpart [1]. For superstrings, the symmetry extends to the Super-Galilean Conformal Algebra, with further differentiation into homogeneous and inhomogeneous variants [4, 7].

The study of tensionless strings provides a novel perspective on addressing several fundamental problems in theoretical physics. First, this regime highlights its connection to higher-spin symmetries, as the disappearance of tension renders all string states massless, allowing higher-spin fields to dominate the spectrum [20]. All these characteristics help to establish links between string theory and Vasiliev's higher-spin theories, as well as with holographic dualities that connect gravitational and quantum field descriptions [23].

Lastly, the Hagedorn transitions represent another key application of the tensionless regime. At temperatures near the Hagedorn temperature, the effective tension of the string decreases to nearly zero, making the tensionless string theory an essential framework for describing this critical phase. In this phase, long strings emerge as the dominant objects, offering a natural explanation for the breakdown of the usual perturbative description of string theory and enabling the study of associated phase transitions [3, 2]. The tensionless limit, therefore, not only provides a window into exploring ultra-relativistic aspects of string theory but also establishes profound connections between symmetries, dualities, and thermal phenomena, opening new possibilities for understanding the fundamental structure of the theory.

The quantum regime of tensionless string theory is crucial for improving our understanding of the phenomena at very high energies. In the tensionless limit, superstring theory seeks to capture these highly energetic phenomena and generalize them to include fermions alongside bosons.

At the heart of quantum string theory, vacua play a central role in shaping and

defining the state space. Specifically, the vacua that emerge from the free open string are tied to the conditions at the free ends of these strings. Similarly, the vacua of closed strings are linked to the absence of free ends. In the tensile theory, the vacua of free closed strings and free open strings are not related. However, under the tensionless limit, closed string vacua become connected to open string vacua, as the closed string behaves like a quantum open string. Furthermore, it has been demonstrated that, in the tensionless regime, multiple distinct types of vacua exist for the closed bosonic string, each leading to different quantum theories [5]. This underscores the sensitivity of quantization to the methods used to impose constraints, offering new insights into the physics of string theory.

Vacua also could be used to define a novel way to define metrics for calculating distances, such as the quantum information metric, which quantify the distances between different theories in a continuous space of quantum field theories. These metrics are essential for studying how theories are related and how they transform into one another through their vacua. [22, 21].

This thesis focuses on exploring the classical and quantum aspects of both tensionless bosonic and tensionless superstrings with focus on the construction of the vacua of the theories. It begins with a treatment of the tensile bosonic string in chapter 1, examining its classical dynamics through the Nambu-Goto and Polyakov actions, and extending to its quantization via light-cone and covariant methods. The chapter 2 turns to the tensile superstring, emphasizing the role of supersymmetry in extending the bosonic string framework. Here we provide a detailed analysis of classical descriptions, including boundary conditions and, the covariant quantization in the (NS,NS) sector.

Chapters 3 and 4 focus on the tensionless regime, where strings lose their tension, revealing new symmetries and physical behaviors. In Chapter 3, the tensionless bosonic string is introduced, highlighting its unique equations of motion, mode expansions, and the emergence of the 2D Galilean Conformal Algebra (GCA). The chapter also investigates the quantum structure, including constraints, induced representations, and the vacuum structure, which notably demonstrates the transformation of closed strings into open strings in this limit.

Chapter 4 extends these concepts to the tensionless superstring, characterized by the Super Galilean Conformal Algebra (SGCA). The classical description includes symmetries, mode expansions, and connections to the tensile limit, while the quantum analysis explores Bogoliubov transformations, vacuum structures, and the interplay between bosonic and fermionic modes. Finally, the last section, will be dedicated to presenting the conclusions and future directions and new results.

Chapter 1

The Bosonic String

In this chapter, we will examine the classical and quantum relativistic bosonic string theory in the Minkowski background space-time. We start by introducing the Nambu-Goto and the Polyakov actions and we describe their respective symmetries. Then, we will use these symmetries to simplify the equations of motion for the open and the closed string models. The quantization process is then presented. It is carried out using both light-cone and covariant methods, both of them bring different insights and provide different tools for handling the physical constraints imposed by Lorentz invariance and reparameterization symmetry. In both methods, the quantum vacuum state and its associated Fock space form a basis for the theory, with excited states arising from the application of string oscillators. Finally, the introduction of boundary states enforce the use of Neumann boundary conditions in the quantum regime.

1.1 Classical Description

1.1.1 The Nambu-Goto Action

The fundamental idea of string theory is to promote the point particle to a relativistic vibrating string with tension T (Fig. 1.1). As it moves through space-time of $D = d + 1$ dimensions, the string draws a two dimensional surface Σ . This surface can be parameterized using two coordinates: $\sigma^0 = \tau$ and $\sigma^1 = \sigma$. The action for a moving string with coordinates $X^\mu(\tau, \sigma)$, where $\mu = 0, \dots, D$, in the Minkowski space is given by

$$\begin{aligned} S_{NG}[X^\mu] &= -T \int_{\Sigma} d\sigma^2 \mathcal{L}, \\ &= -T \int_{\Sigma} d\sigma^2 \sqrt{-\det(\gamma_{\alpha\beta})}, \quad \gamma_{\alpha\beta} = \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \eta_{\mu\nu}, \\ &= -T \int_{\Sigma} d\sigma^2 \sqrt{-\dot{X}^2 X'^2 + (\dot{X} X')^2}. \end{aligned} \tag{1.1}$$

Here, $\partial_{\tau} X^{\mu} = \dot{X}^{\mu}$, $\partial_{\sigma} X^{\mu} = X'^{\mu}$, and $\gamma_{\alpha\beta}$ is the induced metric from a D -dimensional Minkowski space. The action (1.1) is called the Nambu-Goto action.

At each point of the world-sheet there is one time-like and one space-like tangent vector [9]. In order to prove that the quantity under the square root in (1.1) is positive let's consider a generic tangent vector $v^{\mu}(\lambda) = \dot{X}^{\mu} + \lambda X'^{\mu}$, where $\lambda \in$

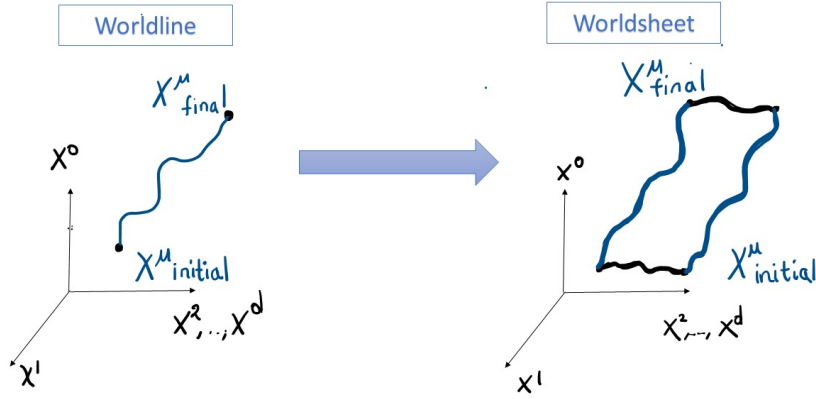


Figure 1.1: From worldlines to worldsheets

$(-\infty, \infty)$ is a parameter. To have both timelike and spacelike tangent vectors, $v^2(\lambda)$ must take both negative and positive values as we vary λ . Then, $v^2(\lambda) = 0$ must have two real roots. Therefore, the discriminant of $v^2(\lambda) = 0$ must be positive, and then

$$(\dot{X} X')^2 - \dot{X}^2 X'^2 > 0. \quad (1.2)$$

The Nambu-Goto action has two types of symmetries: The global and the local ones. Thus, the **global symmetry** corresponds to the *Poincare invariance*, a symmetry on the worldsheet:

$$X^\mu \rightarrow \tilde{X}^\mu = \Lambda^\mu_\nu X^\nu + b^\mu \quad (1.3)$$

Under this transformation, the elements of the induced metric in (1.1) change as

$$\gamma_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu = \Lambda^\mu_\gamma \eta_{\mu\nu} \Lambda^\nu_\sigma \partial_\alpha X^\gamma \partial_\beta X^\sigma = \eta_{\gamma\sigma} \partial_\alpha X^\gamma \partial_\beta X^\sigma. \quad (1.4)$$

Therefore,

$$\begin{aligned} \tilde{S}_{NG} &= -T \int d\sigma^2 \sqrt{-\det(\tilde{\gamma}_{\alpha\beta})}, \quad \tilde{\gamma}_{\alpha\beta} = \partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \eta_{\mu\nu}, \\ &= -T \int_\Sigma d\sigma^2 \sqrt{-\det(\gamma_{\alpha\beta})}, \quad \gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \\ &= S_{NG}. \end{aligned} \quad (1.5)$$

The **local symmetry** called *diffeomorphism invariance* is a gauge symmetry on the worldsheet. We may redefine the worldsheet coordinates as

$$\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha(\tau, \sigma) \quad (1.6)$$

Under this transformation, the line element $ds^2 = \gamma_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = \tilde{\gamma}_{ab} d\tilde{\sigma}^a d\tilde{\sigma}^b$ remains invariant. Then, $\gamma_{\alpha\beta} = (\tilde{M}^T)_{\alpha a} \tilde{\gamma}_{ab} \tilde{M}_b^\beta$, where $\tilde{M}_\alpha^\beta = \partial_\alpha \tilde{\sigma}^\beta$

$$\begin{aligned} S_{NG}[X^\mu] &= \int d\sigma^2 \sqrt{-\det(\gamma_{\alpha\beta})} = \int d\tilde{\sigma}^2 |\det(M)| \sqrt{-\det(\tilde{M}^T) \det(\tilde{\gamma}_{\alpha\beta}) \det(\tilde{M})}, \\ &= \int d\tilde{\sigma}^2 |\det(M)| |\det(\tilde{M})| \sqrt{-\det(\tilde{\gamma}_{\alpha\beta})} = \tilde{S}_{NG} \end{aligned} \quad (1.7)$$

1.1.2 Equations of Motion and Boundary Conditions

In order to obtain the equations of motion, we vary the Nambu-Goto action (1.1) in the following way:

$$\begin{aligned}\delta S_{NG} &= \int d\sigma^2 \left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \partial_\tau \delta X^\mu + \frac{\partial \mathcal{L}}{\partial X'^\mu} \partial_\sigma \delta X^\mu \right], \\ &= \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma [\partial_\tau (\delta X^\mu \mathcal{P}_\mu) + \partial_\sigma (\delta X^\mu \Pi_\mu) - \delta X^\mu (\partial_\tau \mathcal{P}_\mu + \partial_\sigma \Pi_\mu)], \quad (1.8) \\ &= \int_{\tau_i}^{\tau_f} d\tau [\delta X^\mu \Pi_\mu]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \delta X^\mu (\partial_\tau \mathcal{P}_\mu + \partial_\sigma \Pi_\mu) = 0,\end{aligned}$$

where the canonical momentum density $\mathcal{P}_\mu$ and Π_μ are given by

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T \frac{(\dot{X} \cdot X') X'_\mu - X'^2 \dot{X}_\mu}{\sqrt{(X' \cdot \dot{X})^2 - \dot{X}^2 X'^2}}, \quad \Pi_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} = -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - \dot{X}^2 X'_\mu}{\sqrt{(X' \cdot \dot{X})^2 - \dot{X}^2 X'^2}}. \quad (1.9)$$

As it can be seen, it is necessary to impose some conditions that the boundary terms in (1.8) need to satisfy. The boundary conditions will provide unique configurations to the strings. The boundary conditions can be classified as:

-*Dirichlet boundary condition*:

$$\partial_\tau X^\mu(\tau, \sigma_1) = 0, \quad \mu \neq 0 \quad (1.10)$$

The Dirichlet boundary condition enforce that the endpoints of the open string remains fixed throughout the motion.

-*Neumann boundary condition*

$$\Pi_\mu(\tau, \sigma_1) = 0 \quad (1.11)$$

This condition does not impose any constraint on the variation of the string coordinate at the endpoints, as can be seen in the term $[\delta X^\mu \Pi_\mu]_0^{\sigma_1}$. Thus, the endpoints are free to move in the space-time.

-*Periodicity condition*

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) \quad (1.12)$$

This condition sets up the configuration of a closed string. In this case, the string coordinates $X^\mu(\tau, \sigma)$ are periodic functions.

With the vanishing of the boundary terms, only the second term on the right-hand side remains in the equation (1.8). This term must vanish for all δX^μ , then we conclude that

$$\partial_\tau \mathcal{P}_\mu + \partial_\sigma \Pi_\mu = 0. \quad (1.13)$$

There are only two constraints that can be obtained by consider the momentum density (1.9) [11]:

$$\mathcal{P} \cdot X' = 0, \quad \mathcal{P}^2 + T^2 X'^2 = 0. \quad (1.14)$$

In order to simplify the equation of motion and the two constrain equations, a class of gauges are introduced, they fix the parameterization of the worldsheet. The parameterizations of open and closed strings are defined as [26]

$$\begin{aligned}n \cdot X(\tau, \sigma) &= \lambda_1 \tau, \\ n \cdot \mathcal{P} &= \lambda_2 n \cdot p,\end{aligned} \quad (1.15)$$

where $\lambda_{1,2}$ are constants that can be determined by integration over the string: $\lambda_1 = \beta\alpha'(n \cdot p)\tau$, $\lambda_2 = \beta/2\pi$ with $\beta = 2$ (open string), 1 (closed string). In (1.15), X^μ are points that lie both on the world-sheet and on the hyperplane with normal vector n^μ (Fig.1.2). In this parameterization the momentum density $n \cdot \mathcal{P}$ is constant so the σ value assigned to a point is proportional to the amount of $n \cdot p$ momentum carried by the portion of string from the endpoint $\sigma = 0$ to the sigma point $\sigma = \sigma_1$ [26].

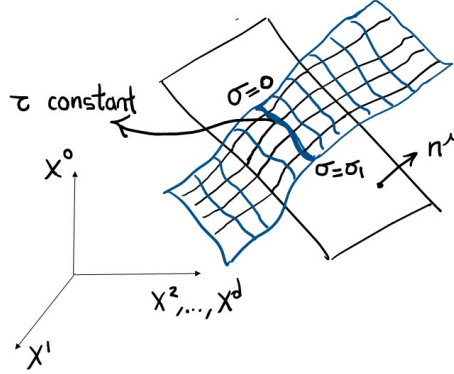


Figure 1.2: String parameterization: the gauge condition $n \cdot X = \lambda\tau$ fixes the string coordinates to be the curve at the intersection of the hyperplane.

In addition, there is (up to scaling) a unique tangent vector which at the same time is orthogonal to the vector X'^μ for every point of the worldsheet. That is,

$$\dot{X} \cdot X' = 0. \quad (1.16)$$

Equation (1.16) is a constraint that follows from the previous parameterization. An additional constraint equation arises when the relations (1.15) and momentum in (1.9) are considered:

$$\mathcal{P}^\mu = \frac{1}{2\pi\alpha'} \frac{X'^2 \dot{X}^\mu}{\sqrt{-\dot{X}^2 X'^2}}, \quad (1.17)$$

$$n \cdot p = \frac{1}{\beta\alpha'} \frac{X'^2 (n \cdot \dot{X})}{\sqrt{-\dot{X}^2 X'^2}}$$

$$\dot{X}^2 + X'^2 = 0 \quad (1.18)$$

Given the above constraints (1.16) and (1.18), the momentum densities $\mathcal{P}^\mu$ and Π^μ simplify considerably:

$$\mathcal{P}^\mu = \frac{1}{2\pi\alpha'} \dot{X}^\mu, \quad \Pi^\mu = -\frac{1}{2\pi\alpha'} X'^{\mu'}. \quad (1.19)$$

Therefore, the equation of motion (1.13) also simplifies to the wave equation:

$$\partial_\tau \mathcal{P}_\mu + \partial_\sigma \Pi_\mu = 0 \quad (1.20)$$

$$\Rightarrow \ddot{X}^\mu - X'^{\mu''} = 0. \quad (1.21)$$

1.1.3 Polyakov Action

The Nambu-Goto action contains a square root which is difficult to treat at the moment of the quantization procedure. It is possible to avoid the square root at the expense of introducing an intrinsic metric tensor $h^{\alpha\beta}$ on the worldsheet. The Polyakov action incorporates this metric tensor to produce an equivalent classical action [9].

$$S[X^\mu, h^{\alpha\beta}] = -\frac{T}{2} \int_{\Sigma} d\sigma^2 \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \quad h = \det(h_{\alpha\beta}). \quad (1.22)$$

Let us now describe the symmetries of the Polyakov action. This action has a **global symmetry**: *Poincare invariance*

$$X^\mu \rightarrow \tilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + b^\mu \quad (1.23)$$

$$\delta X^\mu = a^\mu{}_\nu X^\nu, \quad a^\mu{}_\nu = -a_\nu{}^\mu \quad (1.24)$$

$$\begin{aligned} \tilde{S} &= -\frac{T}{2} \int d\sigma^2 \sqrt{-h} h^{\alpha\beta} \partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \eta_{\mu\nu}, \\ &= -\frac{T}{2} \int d\sigma^2 \sqrt{-h} h^{\alpha\beta} (\partial_\alpha X^\mu + a^\mu{}_\gamma \partial_\alpha X^\gamma) (\partial_\beta X^\nu + a^\nu{}_\delta \partial_\beta X^\delta) \eta_{\mu\nu}, \\ &= S - \frac{T}{2} \int d\sigma^2 \sqrt{-h} h^{\alpha\beta} (\partial_\alpha X^\mu \partial_\beta X^\delta a^\nu{}_\delta + \partial_\alpha X^\gamma \partial_\beta X^\nu a^\mu{}_\gamma) \eta_{\mu\nu}, \\ &= S - \frac{T}{2} \int d\sigma^2 \sqrt{-h} h^{\alpha\beta} (\partial_\alpha X^\mu \partial_\beta X^\delta a_{\mu\delta} + \partial_\alpha X^\gamma \partial_\beta X^\nu a_{\nu\gamma}), \\ &= S - \frac{T}{2} \int d\sigma^2 \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu (a_{\mu\nu} + a_{\nu\mu}), \\ &= S. \end{aligned} \quad (1.25)$$

Moreover, the Polyakov action has two **local symmetries**, the first one is *Diffeomorphism invariance*:

$$\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha(\tau, \sigma) \quad (1.26)$$

Under the transformation (1.26) the metric tensor changes in the following way:

$$h^{\alpha\beta}(\sigma) \rightarrow \tilde{h}^{\alpha\beta}(\tilde{\sigma}) = h^{\gamma\delta}(\sigma) \frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\gamma} \frac{\partial \tilde{\sigma}^\beta}{\partial \sigma^\delta} \quad (1.27)$$

$$\begin{aligned} \tilde{S} &= -\frac{T}{2} \int d\tilde{\sigma}^2 \sqrt{-\tilde{h}} \tilde{h}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \\ &= -\frac{T}{2} \int d\sigma^2 \tilde{J} \sqrt{-h} \tilde{J}^{-2} h^{ab} \frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^a} \frac{\partial \tilde{\sigma}^\beta}{\partial \sigma^b} \frac{\partial X^\mu}{\partial \tilde{\sigma}^\alpha} \frac{\partial X^\nu}{\partial \tilde{\sigma}^\beta} \eta_{\mu\nu}, \\ &= -\frac{T}{2} \int d\sigma^2 \sqrt{-h} h^{ab} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \eta_{\mu\nu}, \quad \tilde{J} = \left| \det \left(\frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\beta} \right) \right| \\ &= S \end{aligned} \quad (1.28)$$

The other one is *Weyl invariance*:

$$h^{\alpha\beta}(\sigma) \rightarrow \tilde{h}^{\alpha\beta} = e^{\phi(\sigma)} h^{\alpha\beta}(\sigma) \quad (1.29)$$

$$\begin{aligned}
\tilde{S} &= -\frac{T}{2} \int d\sigma^2 \sqrt{-\tilde{h}} \tilde{h}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \\
&= -\frac{T}{2} \int d\sigma^2 \sqrt{-e^{2\phi} h} e^\phi h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \\
&= S.
\end{aligned} \tag{1.30}$$

The Weyl invariance can be thought as the invariance of the theory under a local change of scale which preserves the angles.

1.1.4 Equations of Motion and Mode Expansion

Now, let's use the condition $\delta S = 0$ to find the equations of motion for the fields X^μ and $h^{\alpha\beta}$:

$$\begin{aligned}
\delta S &= \delta S_X + \delta S_h, \\
\delta S_X &= -T \int \sigma^2 \sqrt{-h} h^{\alpha\beta} \partial_\alpha (\delta X^\mu) \partial_\beta X^\nu \eta_{\mu\nu}, \\
&= T \int d\sigma^2 \partial_\alpha (\sqrt{-h} h^{\alpha\beta} \partial_\beta X^\nu \eta_{\mu\nu}) \delta X^\mu,
\end{aligned} \tag{1.31}$$

where we have used integration by parts and boundary conditions on the variation δX^μ to suppress the term $\sqrt{-h} h^{\alpha\beta} \partial_\beta X^\nu \eta_{\mu\nu} \delta X^\mu|_{B.C.} = 0$ in the term δS_X . On the other hand, the variation δS_h is such that

$$\begin{aligned}
\delta S_h &= -\frac{T}{2} \int d\sigma^2 \delta(\sqrt{-h}) h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \sqrt{-h} \delta h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \\
\delta h &= h \text{Tr}(h^{\alpha\beta} \delta h_{\alpha\beta}) = h h^{\alpha\beta} \delta h_{\alpha\beta} = -h h_{\alpha\beta} \delta h^{\alpha\beta}, \quad \delta(\sqrt{-h}) = -\frac{\delta h}{2\sqrt{-h}}.
\end{aligned} \tag{1.32}$$

$$\delta S_h = -\frac{T}{2} \int d\sigma^2 \sqrt{-h} \delta h^{\alpha\beta} (\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{h_{\alpha\beta}}{2} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu}) \tag{1.33}$$

From the variations (1.31) and (1.33) the equation of motion can be obtained:

$$\partial_\alpha (\sqrt{-h} h^{\alpha\beta} \partial_\beta X^\mu) = 0, \tag{1.34}$$

$$\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = 0. \tag{1.35}$$

At this point, we can see the classical equivalence between the Nambu-Goto and Polyakov actions by consider the determinant and the square root of the equation of motion (1.35):

$$\sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})} = \frac{1}{2} \sqrt{-h} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} \tag{1.36}$$

Therefore, we have that

$$S = -T \int d\sigma^2 \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})} = -\frac{T}{2} \int d\sigma^2 \sqrt{-h} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} \tag{1.37}$$

In order to simplify the equations of motion (1.34) and (1.35), the world-sheet metric $h_{\alpha\beta}$ can be set to be proportional to the two-dimensional Minkowski metric

$\eta_{\alpha\beta}$, which defines distances as $-ds^2 = -d\tau^2 + d\sigma^2$, as can be seen in the appendix B. Therefore, the equations (1.34) and (1.35) translate to

$$\partial_\alpha \left(\sqrt{-h} h^{\alpha\beta} \partial_\beta X^\mu \right) = \eta^{\alpha\beta} \partial_\alpha \partial_\beta X^\mu = \ddot{X}^\mu - X^{\mu''} = 0 \quad (1.38)$$

$$\begin{aligned} T_{\alpha\beta} &= \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = 0, \\ T_{\alpha\beta} &= \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} \eta_{\alpha\beta} (-\dot{X}^2 + X'^2) = 0, \end{aligned} \quad (1.39)$$

where $T_{\alpha\beta}$ can be identified as the energy-momentum tensor. For $\alpha = \beta = 0$ and $\alpha = 0, \beta = 1$, the equation (1.39) splits out in two constrain equations:

$$\dot{X}^2 + X'^2 = 0, \quad \dot{X} \cdot X' = 0. \quad (1.40)$$

Other combinations of α, β give the same equations (1.40). With these equations, we recover the classical results (1.16), (1.18) and (1.20) from the Nambu-Goto action.

Closed String

The general solution of the wave equation, and the periodicity condition gives

$$\begin{aligned} X^\mu(\tau, \sigma) &= X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma) = X_L^\mu(u) + X_R^\mu(v), \\ X_L^\mu(u + 2\pi) - X_L^\mu(u) &= X_R^\mu(v) - X_R^\mu(v - 2\pi) \end{aligned} \quad (1.41)$$

As a consequence of (1.41), we find that both $X_L^\mu(u)$ and $X_R^\mu(v)$ are strictly periodic functions with period 2π . Therefore,

$$\begin{aligned} X_L^\mu(u) &= \frac{1}{2} x_L^\mu + \sqrt{\frac{\alpha'}{2}} \bar{\alpha}_0 u + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-inu}, \\ X_R^\mu(v) &= \frac{1}{2} x_R^\mu + \sqrt{\frac{\alpha'}{2}} \alpha_0 v + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-inv}. \end{aligned} \quad (1.42)$$

A set of barred α modes was introduced for the expansion of $X_L^\mu(u)$. Even though they are written identically, the unbarred α modes used in the expansion of $X_R^\mu(v)$ have no relation to the open string modes. In closed string theory we need two sets of α modes, barred and unbarred since the left- and right-moving components are completely independent. Then, the full expansion of X^μ in arbitrary Fourier modes $\alpha_n^\mu, \bar{\alpha}_m^\mu$ is given by

$$X^\mu(\tau, \sigma) = x^\mu + \sqrt{2\alpha'} \alpha_0^\mu \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^\mu e^{-in(\tau+\sigma)} + \bar{\alpha}_n^\mu e^{-in(\tau-\sigma)}], \quad (1.43)$$

where $x^\mu = (x_L^\mu + x_R^\mu)/2$, and the constants have been chosen for later convenience. We can identify $q^\mu(0) = x^\mu$ as the 'center of mass position' of the string at $\tau = 0$:

$$q^\mu(\tau) = \frac{1}{2\pi} \int_0^{2\pi} d\sigma X^\mu = x^\mu + \sqrt{2\alpha'} \alpha_0^\mu \tau. \quad (1.44)$$

In addition, the zero Fourier mode α_0 can be related with the total momentum p^μ of the string in the following way

$$p^\mu = \int_0^{2\pi} d\sigma \mathcal{P}^\mu = \frac{1}{2\pi\alpha'} \int_0^{2\pi} \sqrt{2\alpha'} \alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu. \quad (1.45)$$

The requirement that $X^\mu(\tau, \sigma)$ be a real function implies that x^μ and p^μ are real and

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^*, \quad \bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^*. \quad (1.46)$$

In order to find the Poisson brackets of the Fourier modes defined in the expansion (1.43), we need to note that the string coordinates X^μ and the momentum density $\mathcal{P}^\mu$ are phase-space variables which satisfy the following Poisson brackets [11]

$$\begin{aligned} \{X^\mu(\sigma, \tau), X^\nu(\sigma', \tau)\} &= \{\mathcal{P}^\mu(\sigma, \tau), \mathcal{P}^\nu(\sigma', \tau)\} = 0, \\ \{X^\mu(\sigma, \tau), \mathcal{P}^\nu(\sigma', \tau)\} &= \eta^{\mu\nu} \delta(\sigma - \sigma'). \end{aligned} \quad (1.47)$$

Using the Poisson brackets (1.47), we can derive the brackets for the Fourier modes (1.46), and we obtain

$$\begin{aligned} \{\alpha_m^\mu, \alpha_n^\nu\} &= \{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\} = -im\delta_{m+n}\eta^{\mu\nu}, \\ \{\bar{\alpha}_m^\mu, \alpha_n^\nu\} &= 0 \end{aligned} \quad (1.48)$$

The constraint equations (1.40) can be written in terms of the oscillator modes (1.48) by introducing the so called Virasoro constraints $\mathcal{L}_m, \bar{\mathcal{L}}_n$ in the conformal gauge at $\tau = 0$, thus

$$\begin{aligned} \mathcal{L}_n &= T \int_0^{2\pi} d\sigma T_{--} e^{in(\tau-\sigma)}, \quad T_{--} = \frac{T_{00}}{2} - \frac{T_{01}}{2} \\ &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu} = 0, \end{aligned} \quad (1.49)$$

$$\begin{aligned} \bar{\mathcal{L}}_n &= T \int_0^{2\pi} d\sigma T_{++} e^{in(\tau+\sigma)}, \quad T_{++} = \frac{T_{00}}{2} + \frac{T_{01}}{2} \\ &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \bar{\alpha}_m^\mu \bar{\alpha}_{n-m}^\nu \eta_{\mu\nu} = 0. \end{aligned} \quad (1.50)$$

which satisfies the following Poisson brackets

$$\begin{aligned} \{\mathcal{L}_n, \mathcal{L}_m\} &= -i(n-m)\mathcal{L}_{n+m}, \\ \{\bar{\mathcal{L}}_n, \bar{\mathcal{L}}_m\} &= -i(n-m)\bar{\mathcal{L}}_{n+m}, \\ \{\mathcal{L}_n, \bar{\mathcal{L}}_m\} &= 0. \end{aligned} \quad (1.51)$$

Open String

For Neumann boundary conditions, the general solution of the wave equation is given by [24]

$$X^\mu(\tau, \sigma) = x^\mu + \sqrt{2\alpha'} \alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma). \quad (1.52)$$

As in the case of the closed string, x^μ and p^μ are the center of mass position and total space-time momentum of the open string, and the zero mode is $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$. The corresponding Poisson bracket for the oscillator modes are

$$\{\alpha_m^\mu, \alpha_n^\nu\} = -im\delta_{m+n}\eta^{\mu\nu} \quad (1.53)$$

Furthermore, the generator $\mathcal{L}_m$ of the Virasoro algebra satisfies the following bracket

$$\{\mathcal{L}_m, \mathcal{L}_n\} = -i(m-n)\mathcal{L}_{m+n} \quad (1.54)$$

where,

$$\begin{aligned} \mathcal{L}_n &= T \int_0^\pi d\sigma [T_{++}e^{in(\tau+\sigma)} + T_{--}e^{in(\tau-\sigma)}], \\ &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu} = 0. \end{aligned} \quad (1.55)$$

1.2 Quantum Description

This section examines the light-cone and covariant quantization of the bosonic string theory. Light-cone quantization involves selecting a preferred frame by fixing two components of the string's spacetime coordinates, which reduces the constraint equations and physical degrees of freedom. This method simplifies the quantization process by eliminating non-physical modes, enabling direct construction of the physical spectrum. In contrast, covariant quantization maintains manifest Lorentz invariance throughout, allowing the treatment of all spacetime coordinates equivalently. Here, the Virasoro constraints emerge to restrict the Hilbert space to physical states.

1.2.1 Light Cone Quantization

It is possible to choose a gauge in which the constraints equations can be solved explicitly and the theory can be formulated in terms of physical degrees of freedom only, and then quantize the theory via canonical quantization. In order to do so, let's consider light cone coordinates $\sqrt{2}\sigma^\pm = \tau \pm \sigma$ under a reparametrization transformation:

$$\begin{aligned} \sigma^+ &\rightarrow \tilde{\sigma}^+ = f(\sigma^+), \\ \sigma^- &\rightarrow \tilde{\sigma}^- = g(\sigma^-). \end{aligned} \quad (1.56)$$

Therefore, the quadratic line element $ds^2 = -d\tau^2 + d\sigma^2 = -2d\sigma^+d\sigma^-$ transform as follows

$$ds^2 \rightarrow d\tilde{s}^2 = -2d\tilde{\sigma}^+d\tilde{\sigma}^- = -2\partial_+f(\sigma^+)\partial_-g(\sigma^-)d\sigma^+d\sigma^- \quad (1.57)$$

The factor $-2\partial_+f(\sigma^+)\partial_-g(\sigma^-)$ can be drop out using the exponential factor $e^{\phi(\sigma,\tau)}$ from Weyl symmetry. So, we preserve ds^2 by applying reparametrization and Weyl invariance. Now, let's consider the following transformation

$$\begin{aligned} \sigma^+ + \sigma^- &= \sqrt{2}\tau \rightarrow \tilde{\sigma}^+ + \tilde{\sigma}^- = \sqrt{2}\tilde{\tau}, \\ \Rightarrow \tilde{\tau} &= \frac{1}{\sqrt{2}}f(\sigma^+) + \frac{1}{\sqrt{2}}g(\sigma^-) = \frac{1}{\sqrt{2}}f(\tau + \sigma) + \frac{1}{\sqrt{2}}g(\tau - \sigma). \end{aligned} \quad (1.58)$$

If we identify the relation (1.58) with the general solution of the wave equation, then we can relate the coordinates X^0, X^1 with τ and X^+ . That is,

$$X^+ = \left(\frac{1}{\sqrt{2}}X^0 + \frac{1}{\sqrt{2}}X^1 \right) \propto \tau \quad (\text{light cone gauge}) \quad (1.59)$$

Hence,

$$X^+ = \beta \alpha' p^+ \tau, \quad (1.60)$$

where $\beta = 2$ for open strings and $\beta = 1$ for closed strings. Since we know X^+ and X^- from the light cone gauge and the constraint equations, then the independent degrees of freedom are the scalar fields X^I where $I = 2, 3, \dots, 25$.

For the independent fields X^I , we apply the canonical quantization by promoting the Poisson brackets from the classical theory to commutators in the quantum regimen:

$$\{ \quad, \quad \}_{PB} \rightarrow \frac{1}{i} [\quad, \quad]. \quad (1.61)$$

Using the expression (1.47), we obtain the following commutators

$$\begin{aligned} [X^I(\tau, \sigma), \mathcal{P}^J(\tau, \sigma')] &= i\delta(\sigma - \sigma')\eta^{IJ}, \\ [X^I(\tau, \sigma), X^J(\tau, \sigma')] &= [\mathcal{P}^I(\tau, \sigma), \mathcal{P}^J(\tau, \sigma')] = 0 \end{aligned} \quad (1.62)$$

From the commutators (1.62), we can derive the commutators for the oscillator modes to obtain

$$\begin{aligned} [\bar{\alpha}_m^I, \bar{\alpha}_n^J] &= m\delta_{m+n}\eta^{IJ}, & [\alpha_m^I, \alpha_n^J] &= m\eta^{IJ}\delta_{m+n}, & [\bar{\alpha}_m^I, \alpha_n^J] &= 0, & (\text{Closed string}) \\ [\alpha_m^I, \alpha_n^J] &= m\eta^{IJ}\delta_{m+n}, & (\text{Open string}). \end{aligned} \quad (1.63)$$

We can define the operators $\alpha_n^I = a_n^I \sqrt{n}$ and $\alpha_{-n}^I = a_n^{I\dagger} \sqrt{n}$ for $n \geq 1$. Using the operators a_m^I and $a_n^{J\dagger}$ into the commutators (1.63), then the operators $(a_m^I, a_n^{J\dagger})$ satisfy the commutation relations of the canonical annihilation and creation operators of a quantum simple harmonic oscillator. There is a pair of creation and annihilation operators for each value $m \geq 1$ of the mode number and for each transverse light-cone direction I . The same applies for the bar operators. With these creation-annihilation operators, we introduce the ground states or vacuum states $|0, p\rangle$ in the bosonic string theory. Thus, by definition, they are annihilated by all the α_n^I operators:

$$\begin{aligned} \alpha_n^I |0, p\rangle &= 0, & \bar{\alpha}_n^I |0, p\rangle &= 0 & n \geq 1, & (\text{Closed string}) \\ \alpha_n^I |0, p\rangle &= 0, & n \geq 1 & (\text{Open string}). \end{aligned} \quad (1.64)$$

To create states from the vacuum state, we can act on them with creation operators. All the creation operators commute among each other, so the order is irrelevant. To generate all the basis states $|\lambda\rangle$ we must act on the ground states with the same amount of creation operators $\alpha_n^{I\dagger}$ and $\alpha_m^{J\dagger}$. Any basis state $|\lambda\rangle$ can be written as [24]

$$\begin{aligned} |\lambda, \bar{\lambda}\rangle &= \left[\prod_{n=1}^{\infty} \prod_{I=2}^D (\alpha_n^{I\dagger})^{\lambda_{n,I}} \right] \times \left[\prod_{m=1}^{\infty} \prod_{J=2}^D (\bar{\alpha}_m^{J\dagger})^{\bar{\lambda}_{m,J}} \right] |0, p\rangle, & (\text{Closed string}) \\ |\lambda\rangle &= \prod_{n=1}^{\infty} \prod_{I=2}^D (\alpha_n^{I\dagger})^{\lambda_{n,I}} |0, p\rangle, & (\text{Open string}). \end{aligned} \quad (1.65)$$

Each state $|\lambda\rangle$ of the quantum string represents a one-particle state of fixed momentum. Different oscillation modes of unique type of string correspond to different kinds of particles [14].

Because of ordering ambiguities in creation-annihilation operators, the classical Virasoro algebra needs to be modified. The quantum Virasoro algebra takes into account these ambiguities by introducing a constant D (dimension), so that the algebra is [9] :

Open String

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n} + \frac{D-2}{12}m(m^2 - 1)\delta_{m+n} \quad (1.66)$$

Closed string

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m - n)\mathcal{L}_{m+n} + \frac{D-2}{12}m(m^2 - 1)\delta_{m+n}, \\ [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] &= (m - n)\bar{\mathcal{L}}_{m+n} + \frac{D-2}{12}m(m^2 - 1)\delta_{m+n}, \\ [\mathcal{L}_m, \bar{\mathcal{L}}_n] &= 0. \end{aligned} \quad (1.67)$$

Finally, the procedure of light-cone quantization allows to solve the Virasoro constraints explicitly, but at the price of losing manifest Lorentz invariance. To check the Lorentz symmetry, the generators must obey the commutation rules of the Lorentz algebra. In order to ensure this invariance, the number of dimensions must be $D = 25 + 1$ [11].

1.2.2 Covariant Quantization

In the covariant quantization, we treat all string coordinates $X^\mu(\tau, \sigma)$ on the same footing to maintain the Lorentz invariance. So, we replace the Poisson brackets of the fundamental phase space variables by commutators for every coordinates:

$$\begin{aligned} [X^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] &= [\mathcal{P}^\mu(\sigma, \tau), \mathcal{P}^\nu(\sigma', \tau)] = 0, \\ [X^\mu(\sigma, \tau), \mathcal{P}^\nu(\sigma', \tau)] &= i\eta^{\mu\nu}\delta(\sigma - \sigma') \end{aligned} \quad (1.68)$$

Using (1.68), the commutation relations for the oscillator modes can be found:

$$\begin{aligned} [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] &= m\delta_{m+n}\eta^{\mu\nu}, & [\alpha_m^\mu, \alpha_n^\nu] &= m\eta^{\mu\nu}\delta_{m+n}, & [\bar{\alpha}_m^\mu, \alpha_n^\nu] &= 0, & \text{(Closed string)} \\ [\alpha_m^\mu, \alpha_n^\nu] &= m\eta^{\mu\nu}\delta_{m+n}, & & & & & \text{(Open string)}. \end{aligned} \quad (1.69)$$

As in the light cone quantization, we introduce the vacuum states for the covariant approach

$$\begin{aligned} \alpha_n^\mu |0, p\rangle &= 0, & \bar{\alpha}_n^\mu |0, p\rangle &= 0, & \text{(Closed string)} \\ \alpha_n^\mu |0, p\rangle &= 0, & & & \text{(Open string)}. \end{aligned} \quad (1.70)$$

Furthermore, the whole Hilbert space of states is obtained by acting on the ground state with creation operators. However, there exist states with negative norm or “ghosts” such as $\langle 0, p | \alpha_m^0 \alpha_m^{0\dagger} | 0, p \rangle = -m < 0$. They do not allow for probability interpretation of the corresponding physical system [9].

In order to define the subspace of physical states in the original Hilbert space, it is necessary to impose some constraints. We promote the classical constraints $\mathcal{L}_m = \bar{\mathcal{L}}_m = 0$ to operators in the quantum regime. They are quadratic in oscillators and might involve operators which do not commute with each other. From all $\mathcal{L}_m$ constraints, $\mathcal{L}_0, \bar{\mathcal{L}}_0$ are the only one which suffer ordering ambiguity [11].

We can see that not all the quantum constraints $\mathcal{L}_m, \bar{\mathcal{L}}_n$ can be imposed on the states. That is, if we try to impose the following constraints

$$\begin{aligned} \mathcal{L}_n |\psi\rangle &= 0 = \bar{\mathcal{L}}_n |\psi\rangle, \quad \forall n \in \mathbb{Z} && \text{(Closed string)} \\ \mathcal{L}_n |\psi\rangle &= 0, \quad \forall n \in \mathbb{Z} && \text{(Open string)} \end{aligned} \quad (1.71)$$

then, for example for the case of the open string

$$\langle \psi | [\mathcal{L}_n, \mathcal{L}_{-n}] | \psi \rangle = \langle \psi | 2n\mathcal{L}_0 | \psi \rangle + \frac{D}{12}n(n^2 - 1) \langle \psi | \psi \rangle. \quad (1.72)$$

This is not possible to satisfy unless $|\psi\rangle = 0$. This confirms that we cannot impose all the Virasoro conditions and expect to find nontrivial states. From the experience with quantum electrodynamics we impose only half of the constraints on physical states [9]:

Closed string

$$\begin{aligned} \mathcal{L}_n |\psi\rangle &= \bar{\mathcal{L}}_n |\psi\rangle = 0, && n \geq 1, \\ (\mathcal{L}_0 - a) |\psi\rangle &= 0, && (\bar{\mathcal{L}}_0 - \bar{a}) |\psi\rangle = 0. \end{aligned} \quad (1.73)$$

Open String

$$\begin{aligned} \mathcal{L}_n |\psi\rangle &= 0, && n \geq 1 \\ (\mathcal{L}_0 - a) |\psi\rangle &= 0, \end{aligned} \quad (1.74)$$

where the constants $a, \bar{a}$ are required to take into account the ambiguity ordering in the zero Virasoro constraint. For space-time dimension $D \leq 26$, the states $|\psi\rangle$ which satisfy the relations (1.73) and (1.74) have positive norm (no-ghost theorem) [24]

Furthermore, the quantum Virasoro constraints $\mathcal{L}_m, \bar{\mathcal{L}}_n$ are the generators of the quantum Virasoro algebra. Using the commutation relations of creation-annihilation operators (1.69), and the normal ordering constants $a, \bar{a}$ for the zero mode, then the Virasoro constraints satisfy the following commutation relations [24]

Open String

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n} + \frac{D}{12}m(m^2 - 1)\delta_{m+n} \quad (1.75)$$

Closed string

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m - n)\mathcal{L}_{m+n} + \frac{D}{12}m(m^2 - 1)\delta_{m+n}, \\ [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] &= (m - n)\bar{\mathcal{L}}_{m+n} + \frac{D}{12}m(m^2 - 1)\delta_{m+n}, \\ [\mathcal{L}_m, \bar{\mathcal{L}}_n] &= 0. \end{aligned} \quad (1.76)$$

1.2.3 Boundary States

The two-dimensional action for a free boson $X^\mu(\tau, \sigma)$ was given in (1.22). In the conformal gauge $h^{\alpha\beta} = \eta^{\alpha\beta}$ this action takes the following form

$$S = -\frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu = \frac{T}{2} \int d^2\sigma (\dot{X}^2 - X'^2) \quad (1.77)$$

Varying with respect to X^μ such that $\delta X^\mu(\tau_1) = \delta X^\mu(\tau_2) = 0$ we obtain

$$\delta S = T \int d^2\sigma \delta X^\mu (\partial_\sigma^2 - \partial_\tau^2) X_\mu - T \int_{\tau_1}^{\tau_2} d\tau X'_\mu \delta X^\mu|_{\sigma=0}^{\sigma=\pi} \quad (1.78)$$

Because we impose a periodicity condition, the surface term is absent from the closed string. For an open string, the vanishing of the boundary term is achieved when we impose either Dirichlet or Neumann boundary. These conditions were described in (1.10) and (1.11). In the conformal gauge, they translate to

$$\begin{aligned} \partial_\sigma X^\mu|_{\sigma=0,\pi} &= 0, & (\text{Neumann b.c.}) \\ \delta X^\mu|_{\sigma=0,\pi} &= 0 & (\text{Dirichlet b.c.}) \end{aligned} \quad (1.79)$$

In particular, it is possible to express the Neumann boundary condition for the field $X(\sigma, \tau)$ in terms of Laurent modes $j(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} j_n$ with $z = \exp(\tau + i\sigma)$ [10].

$$\partial_\sigma X = i(\partial - \bar{\partial})X = j(z) - \bar{j}(\bar{z}) = \sum_{n \in \mathbb{Z}} (j_n z^{-n-1} - \bar{j}_n \bar{z}^{-n-1}). \quad (1.80)$$

In this way, the Neumann boundary condition is given by

$$\begin{aligned} \partial_\sigma X|_{\sigma=0} &= \sum_{n \in \mathbb{Z}} j_n e^{-n(\tau+i\sigma)} - \bar{j}_n e^{-n(\tau-i\sigma)} = 0, \\ &= \sum_{n \in \mathbb{Z}} (j_n - \bar{j}_n) e^{-n\tau} = 0. \end{aligned} \quad (1.81)$$

Since for generic τ the summands above are linearly independent, for all integers n these two equations are solved by

$$j_n - \bar{j}_n = 0. \quad (1.82)$$

As illustrated in Figure 1.1, by interchanging τ and σ , we can reinterpret the cylinder partition function of the boundary conformal field theory (BCFT) on one side as a tree-level amplitude of the underlying conformal field theory (CFT) on the other side. This reinterpretation highlights the open/closed string duality, where the tree-level amplitude describes the emission of a closed string from boundary A, its propagation, and its absorption at boundary B. In this context, a boundary can be interpreted as an object, which couples to closed strings [10]. The relation above then reads

$$(\tau, \sigma)_{\text{closed}} \leftrightarrow (\sigma, \tau)_{\text{open}} \quad (1.83)$$

With this equivalence relation between open and closed strings, the Neumann boundary condition can be expressed in terms of the fields from the closed string.

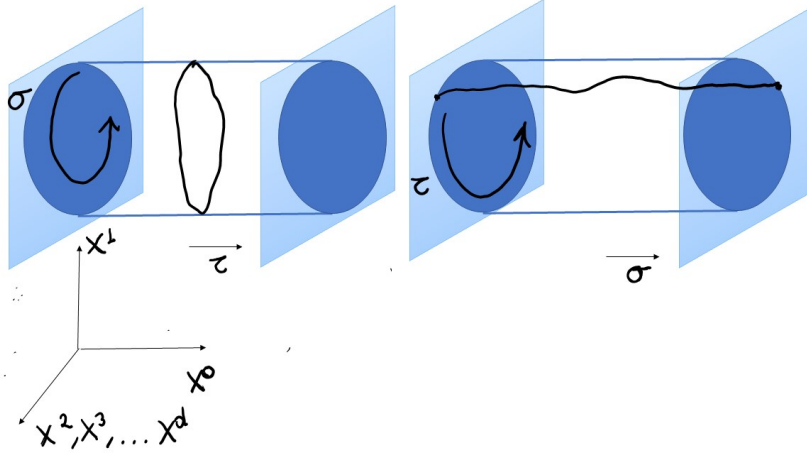


Figure 1.3: Closed/open string duality

Then, the Neumann boundary condition is introduced as a constraint which is satisfied by the so called boundary states $|B_N\rangle$ in the following way:

$$\partial_\tau X_{closed}|_{\tau=0} |B_N\rangle = 0. \quad (1.84)$$

In terms of the Laurent modes, the boundary states are such that

$$i\partial_\tau X_{closed} |B_N\rangle = \sum_{n \in \mathbb{Z}} (j_n e^{-in\sigma} + \tilde{j}_n e^{in\sigma}) |B_N\rangle = 0, \quad \tau = 0 \quad (1.85)$$

$$(j_n + \tilde{j}_{-n}) |B_N\rangle = 0, \quad \forall n \in \mathbb{Z} \quad (1.86)$$

The equation (1.86) is satisfied by the solution [10]

$$|B_N\rangle = \frac{1}{\mathcal{N}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} j_{-k} \bar{j}_{-k} \right) |0\rangle, \quad (1.87)$$

where $\mathcal{N}$ is a normalization constant to be fixed.

Chapter 2

The Superstring

The bosonic string theory discussed in the previous chapter is unsatisfactory because it does not contain fermions. Superstring formalism provides a framework for incorporating fermionic degrees of freedom into string theory, extending the bosonic string model to a supersymmetric context. In the present chapter, we explore this approach that formulates superstrings in 10-dimensional spacetime, where both bosonic and fermionic coordinates coexist on the worldsheet. The theory introduces two sectors, the Ramond and Neveu-Schwarz sectors, which correspond to periodic and antiperiodic boundary conditions on the fermionic coordinates, respectively. Subsequently, the quantization process involves imposing worldsheet supersymmetry constraints and constructing the quantum state space.

2.1 Classical Description

As was discussed in the previous chapter, the Polyakov action (1.22) is invariant under Weyl and diffeomorphism transformations, so that we can map the worldsheet metric $h_{\alpha\beta}$ to the flat metric $\eta_{\alpha\beta}$. In this conformal gauge, the action takes the following expression:

$$S = -\frac{T}{2} \int_{\Sigma} d\sigma^2 \partial_{\alpha} X \cdot \partial^{\alpha} X. \quad (2.1)$$

The action (2.1) describes a free field theory in two dimensions. To generalize this action, we introduce additional internal degrees of freedom describing fermions on the world sheet. The desired action is obtained by introducing the standard Dirac action for two-dimensional Majorana spinor which also has space-time vector index $\mu = 1, 2, \dots, D$ [8]:

$$S[X^{\mu}, \psi^{\mu}] = -\frac{T}{2} \int_{\Sigma} d\sigma^2 (\partial_{\alpha} X \cdot \partial^{\alpha} X - \bar{\psi} \cdot \rho^{\alpha} \partial_{\alpha} \psi) \quad (2.2)$$

Here, ψ^{μ} is a spinor on the two-dimensional world sheet but transform as a vector under spacetime Lorentz transformations from $SO(D-1, 1)$. They are the fermionic counterpart of the scalar X^{μ} . The spinors are two Majorana components denoted by

$$\psi^{\mu} = \begin{pmatrix} \psi_{-}^{\mu} \\ \psi_{+}^{\mu} \end{pmatrix}, \quad \bar{\psi} = \psi^T i \rho^0 \quad (2.3)$$

Furthermore, ρ^α represents the two-dimensional Dirac matrices which obey the Clifford algebra:

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}, \quad (2.4)$$

where ρ^0 and ρ^1 are conventionally chosen as

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.5)$$

2.1.1 Boundary Conditions and Mode Expansions

The equations of motion for the action (2.2) can be obtained from the variation $\delta S = 0$. Since the equations of motion for the fields X^μ were founded in the previous chapter, we will only focus on the variation of the fields ψ^μ . Let us consider the variation of the fermionic part S_F in light-cone coordinates.

$$S_F = \int d\sigma^2 \bar{\psi} \cdot \rho^\alpha \partial_\alpha \psi = \int d\sigma^2 (\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+), \quad (2.6)$$

$$\begin{aligned} \delta S_F &= \int d\sigma^2 (\delta \psi_- \partial_+ \psi_- + \psi_- \partial_+ \delta \psi_- + \delta \psi_+ \partial_- \psi_+ + \psi_+ \partial_- \delta \psi_+), \\ &= \int d\sigma^2 [\delta \psi_- \partial_+ \psi_- + \partial_+ (\psi_- \delta \psi_-) - \partial_+ \psi_- \delta \psi_- + \delta \psi_+ \partial_- \psi_+ \\ &\quad + \partial_- (\psi_+ \delta \psi_+) - \partial_- \psi_+ \delta \psi_+], \end{aligned} \quad (2.7)$$

$$\delta S_F = \int d\sigma^2 [\partial_+ (\psi_- \delta \psi_-) + \partial_- (\psi_+ \delta \psi_+) - 2\partial_+ \psi_- \delta \psi_- - 2\partial_- \psi_+ \delta \psi_+] = 0 \quad (2.8)$$

The equations of motion are obtained from the last two terms in (2.8). They are given by,

$$\partial_+ \psi_-^\mu = 0, \quad \partial_- \psi_+^\mu = 0, \quad (2.9)$$

which implies that $\psi_-^\mu = \psi_-^\mu(\tau - \sigma)$ is a right-moving spinor component while $\psi_+^\mu = \psi_+^\mu(\tau + \sigma)$ is a left-moving spinor component. The vanishing of the remaining terms in (2.8) is determined by the application of boundary conditions. In order to find the conditions, let us expand the surface terms from (2.8)

$$\begin{aligned} &\int d\sigma^2 [\partial_+ (\psi_- \delta \psi_-) + \partial_- (\psi_+ \delta \psi_+)], \\ &= \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma [(\partial_\tau + \partial_\sigma)(\psi_- \delta \psi_-) + (\partial_\tau - \partial_\sigma)(\psi_+ \delta \psi_+)], \\ &= \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau [\psi_- \cdot \delta \psi_- - \psi_+ \cdot \delta \psi_+]_0^{\sigma_1} \end{aligned} \quad (2.10)$$

The fields ψ_+^μ and ψ_-^μ appear quadratically in the action (2.2). Then, the signs of ψ_+^μ and ψ_-^μ can be changed without physical consequence. This arbitrariness is used conventionally to demand in (2.10) the following condition at the endpoint $\sigma = 0$ [26]:

$$\psi_-^\mu(0, \tau) = +\psi_+^\mu(0, \tau) \quad \Rightarrow \quad \delta \psi_-^\mu(\tau, 0) = +\delta \psi_+^\mu(\tau, 0) \quad (2.11)$$

The other boundary condition needs to be imposed according to the different configurations of the string. For an open string, the endpoint is $\sigma_1 = \pi$ and since we cannot change the sign of ψ_+^μ and ψ_-^μ without changing the condition (2.11), the choice of the sign at the endpoint σ_1 is physically relevant. Then, we have to impose two different cases given by

$$\psi_+^\mu(\tau, \pi) = \pm \psi_-^\mu(\tau, \pi) \quad \Rightarrow \quad \delta\psi_+^\mu(\tau, \pi) = \pm \delta\psi_-^\mu(\tau, \pi) \quad (2.12)$$

In this way, the conditions (2.11) and (2.12) vanish the terms shown in (2.10). For the closed string, we must impose periodic boundary conditions, and also we need to demand two different cases such that

$$\begin{aligned} \psi_+^\mu(\tau, \sigma + 2\pi) &= \pm \psi_+^\mu(\tau, \sigma) & \Rightarrow \delta\psi_+^\mu(\tau, \sigma) &= \pm \delta\psi_+^\mu(\tau, \sigma + 2\pi), \\ \psi_-^\mu(\tau, \sigma + 2\pi) &= \pm \psi_-^\mu(\tau, \sigma) & \Rightarrow \delta\psi_-^\mu(\tau, \sigma) &= \pm \delta\psi_-^\mu(\tau, \sigma + 2\pi). \end{aligned} \quad (2.13)$$

The vanishing of the terms (2.10) in the case of the closed string is consequently expressed by the conditions (2.11) and (2.13).

We have seen that for the open string, there are two different ways to impose the boundary conditions. It means that the full superstring theory state space breaks into two subspaces: a Ramond (R) sector which contains the states that arise using the top choice of sign, and a Neveu–Schwarz (NS) sector which contains the states that arise using the lower choice of sign. However, for the closed string, the conditions for the two spinor components ψ_+ and ψ_- can be chosen independently and satisfy $\delta S = 0$, leading to a total of four possibilities: (R,R), (NS,NS), (NS,R) and (R,NS) [9].

-Ramond sector

With Ramond boundary conditions applied to the closed string, the field $\psi_\pm$ exhibits periodicity and can be expanded in terms of integrally moded oscillators.

$$\psi_-^\mu(\sigma^-) = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-in\sigma^-} \quad (2.14)$$

$$\psi_+^\mu(\sigma^+) = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \tilde{d}_n^\mu e^{-in\sigma^+} \quad (2.15)$$

-Neveu-Schwarz sector

In the case of the closed string, since the Neveu–Schwarz fermion $\psi_\pm$ is a function of $\sigma^- = \tau - \sigma$ and changes sign when $\sigma \rightarrow \sigma + 2\pi$, it must be expanded with fractionally moded exponentials:

$$\psi_-^\mu(\sigma^-) = \sqrt{2\alpha'} \sum_{r \in \mathbb{Z} + 1/2} b_r^\mu e^{-ir\sigma^-} \quad (2.16)$$

$$\psi_+^\mu(\sigma^+) = \sqrt{2\alpha'} \sum_{r \in \mathbb{Z} + 1/2} \tilde{b}_r^\mu e^{-ir\sigma^+} \quad (2.17)$$

For the open string, the mode expansions for the fermionic fields in the R and NS sectors are given by only one copy of the closed string mode expansions. That is, (2.14) and (2.16) describe the mode expansions for the R and NS sectors for the open string, respectively.

2.1.2 Symmetries and Constrain Equations

Let us now consider the conserved currents associated with the global symmetries of the action (2.2). The first one is the energy-momentum tensor $T_{\alpha\beta}$ associated with the translation symmetry. The second one is the supercurrent J^α associated with the global world-sheet supersymmetry.

We begin by examining the traslation symmetry. Here, we consider the infinitesimal traslation of world-sheet coordinates $\tilde{\sigma}^\alpha = \sigma^\alpha + a^\alpha$. Then, the fields $X^\mu(\tau, \sigma)$ and $\psi^\mu(\tau, \sigma)$ varies as

$$\begin{aligned}\delta X^\mu &= -a^\alpha \partial_\alpha X^\mu, \\ \delta \psi^\mu &= -a^\alpha \partial_\alpha \psi^\mu.\end{aligned}\tag{2.18}$$

$$\begin{aligned}\delta S &= -\frac{T}{2} \int d\sigma^2 [2\partial^\alpha X^\mu \partial_\alpha(\delta X_\mu) + \delta \bar{\psi} \rho^\alpha \partial_\alpha \psi_\mu + \bar{\psi} \rho^\alpha \partial_\alpha(\delta \psi_\mu)], \\ &= -\frac{T}{2} \int d\sigma^2 [-2\partial^\alpha X^\mu \partial_\alpha(a^\beta \partial_\beta X_\mu) + a^\beta \partial_\alpha \bar{\psi}_\mu \rho^\alpha \partial_\beta \psi^\mu - \bar{\psi}^\mu \rho^\alpha \partial_\alpha(a^\beta \partial_\beta \psi_\mu)], \\ &= \frac{T}{2} \int d\sigma^2 a^\beta \partial_\beta (\partial_\alpha X \cdot \partial^\alpha X + \bar{\psi} \cdot \rho^\alpha \partial_\alpha \psi), \\ &= T \int d\sigma^2 \partial_\beta \mathcal{J}^\beta,\end{aligned}\tag{2.19}$$

where we have identified

$$\mathcal{J}_\beta = \frac{a_\beta}{2} (\partial_\alpha X \cdot \partial^\alpha X + \bar{\psi} \cdot \rho^\alpha \partial_\alpha \psi).\tag{2.20}$$

We calculate the energy-momentum tensor using (2.20), and the Noether current method.

$$\begin{aligned}a^\alpha T_{\alpha\beta} &= \frac{\partial \mathcal{L}}{\partial(\partial^\beta X^\mu)} \delta X^\mu + \frac{\partial \mathcal{L}}{\partial(\partial^\beta \psi^\mu)} \delta \psi^\mu - \mathcal{J}_\beta, \\ &= \partial_\beta X_\mu a^\alpha \partial_\alpha X^\mu + \frac{1}{2} \bar{\psi}_\mu \rho_\beta a^\alpha \partial_\alpha \psi^\mu - \frac{a_\beta}{2} (\partial_\alpha X \cdot \partial^\alpha X + \bar{\psi} \cdot \rho^\alpha \partial_\alpha \psi), \\ &= a^\alpha \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{a^\alpha}{4} (\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu + \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu) \\ &\quad + \frac{a^\alpha}{4} (\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu) - \frac{a^\alpha}{2} \eta_{\alpha\beta} (\partial_\gamma X \cdot \partial^\gamma X + \bar{\psi} \cdot \rho^\gamma \partial_\gamma \psi), \\ &= a^\alpha [\partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{4} \bar{\psi}^\mu \rho_{(\alpha} \partial_{\beta)} \psi_\mu - \frac{\eta_{\alpha\beta}}{2} (\partial_\gamma X \cdot \partial^\gamma X + \bar{\psi} \cdot \rho^\gamma \partial_\gamma \psi) \\ &\quad - \frac{1}{4} \bar{\psi}^\mu \rho_{[\alpha} \partial_{\beta]} \psi_\mu].\end{aligned}\tag{2.21}$$

In (2.21) we have adopted the notation $(\cdot, \cdot)$ for symmetrizing indices, and $[\cdot, \cdot]$ for antisymmetrizing indices. The energy-momentum tensor must be symmetric for indices α, β , thus we drop the antisymmetric part. Therefore,

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{4} \bar{\psi}^\mu \rho_{(\alpha} \partial_{\beta)} \psi_\mu - \frac{\eta_{\alpha\beta}}{2} (\partial_\gamma X \cdot \partial^\gamma X + \bar{\psi} \cdot \rho^\gamma \partial_\gamma \psi).\tag{2.22}$$

Since $T_{\alpha\beta}$ is symmetric and traceless, it has only two independent components. In light cone coordinates, the energy-momentum tensor can be written as [24]

$$\begin{aligned}
T_{+-} = T_{-+} &= \partial_+ X \cdot \partial_- X + \frac{1}{4} \bar{\psi}^\mu (\rho_+ \partial_- + \rho_- \partial_+) \psi_\mu \\
&\quad - \frac{\eta_{+-}}{2} [2\eta^{+-} \partial_+ X \cdot \partial_- X + \eta^{+-} \bar{\psi}^\mu (\rho_+ \partial_- + \rho_- \partial_+) \psi_\mu], \\
&= \partial_+ X \cdot \partial_- X + \frac{1}{4} \bar{\psi}^\mu (\rho_+ \partial_- + \rho_- \partial_+) \psi_\mu \\
&\quad - \partial_+ X \cdot \partial_- X - \frac{1}{2} \bar{\psi}^\mu (\rho_+ \partial_- + \rho_- \partial_+) \psi_\mu, \\
&= -\frac{1}{4} \bar{\psi}^\mu (\rho_+ \partial_- + \rho_- \partial_+) \psi_\mu = 0
\end{aligned} \tag{2.23}$$

As shown in (2.23), the components $T_{+-} = T_{-+} = 0$ automatically vanish due to the equations of motion. The only two independent components are T_{++} and T_{--} :

$$T_{++} = \partial_+ X \cdot \partial_+ X + i \frac{1}{2} \psi_+ \cdot \partial_+ \psi_+, \quad T_{--} = \partial_- X \cdot \partial_- X + i \frac{1}{2} \psi_- \cdot \partial_- \psi_- \tag{2.24}$$

We continue with the super-symmetry. This symmetry ensures that the number of bosonic and fermionic degrees of freedom are the same at any mass level. The infinitesimal super-symmetry transformation is given by

$$\begin{aligned}
\delta X^\mu &= \bar{\epsilon} \psi^\mu, \\
\delta \psi^\mu &= \rho^\alpha \epsilon \partial_\alpha X^\mu,
\end{aligned} \tag{2.25}$$

where $\epsilon = (\epsilon_-, \epsilon_+)$ is a constant infinitesimal Majorana spinor that consists of anti-commuting Grassman numbers. The variation of the action is thus

$$\begin{aligned}
\delta S &= -\frac{T}{2} \int d^2\sigma [2\partial^\alpha X^\mu \partial_\alpha (\bar{\epsilon} \psi_\mu) - \partial_\alpha \bar{\psi}^\mu \rho^\alpha \delta \psi_\mu + \bar{\psi}^\mu \rho^\mu \partial_\alpha \delta \psi_\mu], \\
&= -\frac{T}{2} \int d\sigma^2 [2\partial_\alpha \bar{\psi}^\mu \partial^\alpha X_\mu \epsilon - \partial_\alpha \bar{\psi}^\mu \rho^\alpha \rho^\beta \partial_\beta X_\mu \epsilon + \bar{\psi}^\mu \rho^\alpha \rho^\beta \partial_\alpha \partial_\beta X_\mu \epsilon], \\
&= -\frac{T}{2} \int d\sigma^2 [2\partial_\alpha \bar{\psi}^\mu \partial^\alpha X_\mu \epsilon + 2\bar{\psi}^\mu \rho^\alpha \rho^\beta \partial_\alpha \partial_\beta X_\mu \epsilon - \partial_\alpha \bar{\psi}^\mu \rho^\alpha \rho^\beta \partial_\beta X^\mu \epsilon \\
&\quad - \bar{\psi}^\mu \rho^\alpha \rho^\beta \partial_\alpha \partial_\beta X_\mu \epsilon], \\
&= -\frac{T}{2} \int d\sigma^2 \partial_\alpha [2\bar{\psi}^\mu \epsilon \partial^\alpha X_\mu - \bar{\psi}^\mu \rho^\alpha \rho^\beta \epsilon \partial_\beta X_\mu].
\end{aligned} \tag{2.26}$$

From the last expression in (2.26), we identify $\mathcal{J}^\alpha$ as

$$\mathcal{J}^\alpha = -\bar{\psi}^\mu \epsilon \partial^\alpha X_\mu + \frac{1}{2} \bar{\psi}^\mu \rho^\alpha \rho^\beta \epsilon \partial_\beta X_\mu. \tag{2.27}$$

We calculate the super-current using (2.27), and the Noether current method.

$$\begin{aligned}
\bar{\epsilon} J^\alpha &= \frac{\partial \mathcal{L}}{\partial (\partial_\alpha X^\mu)} \delta X^\mu + \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \psi^\mu)} \delta \psi^\mu - \mathcal{J}^\alpha, \\
&= -\partial^\alpha X_\mu \delta X^\mu - \frac{1}{2} \bar{\psi}_\mu \rho^\alpha \delta \psi^\mu + \bar{\psi}^\mu \epsilon \partial^\alpha X_\mu - \frac{1}{2} \bar{\psi}^\mu \rho^\alpha \rho^\beta \epsilon \partial_\beta X_\mu, \\
&= -\bar{\psi}^\mu \rho^\alpha \rho^\beta \epsilon \partial_\beta X_\mu = -\bar{\epsilon} \rho^\beta \rho^\alpha \psi^\mu \partial_\beta X_\mu
\end{aligned} \tag{2.28}$$

Therefore, the super-current is given by

$$J^\alpha = -\rho^\beta \rho^\alpha \psi^\mu \partial_\beta X_\mu. \quad (2.29)$$

Since $\rho_\alpha \rho^\beta \rho^\alpha = 0$, then $\rho_\alpha J^\alpha = 0$. From this constraint, we see that there are only two independent components of the super current (2.29) [8]. The independent components can be written in light coordinates as follows

$$\begin{aligned} J_+ &= -\rho^\beta \rho_+ \psi^\mu \partial_\beta X_\mu, \\ &= -\rho^+ \rho_+ \psi^\mu \partial_+ X_\mu - \rho^- \rho_+ \psi^\mu \partial_- X_\mu, \\ &= 2(\rho_- \rho_+ \psi^\mu \partial_+ X_\mu + \rho^2 \psi^\mu \partial_- X_\mu), \\ &= 2\rho_- \rho_+ \psi \cdot \partial_+ X, \\ &= 2 \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \cdot \partial_+ X, \\ &= -2 \begin{pmatrix} 0 \\ \psi_+ \cdot \partial_+ X \end{pmatrix} = \begin{pmatrix} 0 \\ -j_+ \end{pmatrix} \end{aligned} \quad (2.30)$$

$$J_- = -2 \begin{pmatrix} \psi_- \cdot \partial_- X \\ 0 \end{pmatrix} = \begin{pmatrix} -j_- \\ 0 \end{pmatrix} \quad (2.31)$$

The requirement of super-conformal symmetry lead to stronger constraints. That is, the vanishing of the energy-momentum tensor and supercurrent [8]:

$$T_{++} = T_{--} = 0, \quad j_+ = j_- = 0. \quad (2.32)$$

We can now expressing, in the (NS,NS) sector, the constrain equations (2.32) in terms of the oscillator modes from (2.14) and (2.16) by introducing the so called super Virasoro constrains for the closed string $\mathcal{L}_n, \bar{\mathcal{L}}_n, \mathcal{Q}_r, \bar{\mathcal{Q}}_r$ and $\mathcal{L}_n, \mathcal{Q}_r$ for the open string [24]:

Closed string

$$\mathcal{L}_n = T \int_0^{2\pi} d\sigma T_{++} e^{in\sigma}, \quad \bar{\mathcal{L}}_n = T \int_0^{2\pi} d\sigma T_{--} e^{-in\sigma} \quad (2.33)$$

$$\mathcal{Q}_r = \frac{T}{\sqrt{2}} \int_0^{2\pi} d\sigma e^{ir\sigma} j_{++}, \quad \bar{\mathcal{Q}}_r = \frac{T}{\sqrt{2}} \int_0^{2\pi} d\sigma e^{-ir\sigma} j_{--} \quad (2.34)$$

In terms of oscillators, the constrains (2.34) become

$$\begin{aligned} \mathcal{L}_n &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{n+m} \cdot \alpha_{-m} + \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} \left(r + \frac{n}{2}\right) b_{-r} \cdot b_{n+r} = 0, \\ \bar{\mathcal{L}}_n &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \bar{\alpha}_{n+m} \cdot \bar{\alpha}_{-m} + \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} \left(r + \frac{n}{2}\right) \bar{b}_{-r} \cdot \bar{b}_{n+r} = 0, \\ \mathcal{Q}_r &= \sum_{m \in \mathbb{Z}} \alpha_m \cdot b_{r-m} = 0, \\ \bar{\mathcal{Q}}_r &= \sum_{m \in \mathbb{Z}} \bar{\alpha}_m \cdot \bar{b}_{r-m} = 0. \end{aligned} \quad (2.35)$$

Open string

$$\mathcal{L}_n = T \int_0^\pi d\sigma (e^{in\sigma} T_{++} + e^{-in\sigma} T_{--}), \quad \mathcal{Q}_r = \frac{T}{\sqrt{2}} \int_0^\pi d\sigma (e^{ir\sigma} j_+ + e^{-ir\sigma} j_-) \quad (2.36)$$

In terms of oscillators, the Virasoro constraints (2.36) become

$$\begin{aligned}\mathcal{L}_n &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{n+m} \cdot \alpha_{-m} + \frac{1}{2} \sum_{r \in \mathbb{Z} + 1/2} \left(r + \frac{n}{2}\right) b_{-r} \cdot b_{n+r} = 0, \\ \mathcal{Q}_r &= \sum_{m \in \mathbb{Z}} \alpha_m \cdot b_{r-m} = 0.\end{aligned}\tag{2.37}$$

Now, let us find the brackets for the fermionic oscillator modes $b_r^\mu, \bar{b}_s^\nu$ from the mode expansions (2.35). We need first the brackets for the fields $\psi_\pm^\mu$. However, since the canonical momentum of $\psi_\pm^\mu$ is $(i/4\pi\alpha')\psi_\pm^\mu$, these constraints are second class, meaning that if we compute their Poisson bracket the right-hand side is not another constraint. Therefore, we must follow the procedure due to Dirac and replace the Poisson brackets by Dirac brackets, which is defined as[9]

$$\{A, B\}_{D.B.} = \{A, B\}_{P.B.} - \{A, \phi_i\}_{P.B.} C_{ij}^{-1} \{\phi_j, B\}_{P.B.}, \tag{2.38}$$

where ϕ_i is a second class constrain and $C_{ij} = \{\phi_i, \phi_j\}_{P.B.}$. This leads to

$$\begin{aligned}\{\psi_\pm^\mu(\tau, \sigma), \psi_\pm^\nu(\tau, \sigma')\}_{D.B.} &= -i\eta^{\mu\nu} 2\pi\alpha' \delta(\sigma - \sigma'), \\ \{\psi_+^\mu(\tau, \sigma), \psi_-^\nu(\tau, \sigma')\}_{D.B.} &= 0.\end{aligned}\tag{2.39}$$

In terms of the oscillators, the brackets (2.39) translate into the relations [9]

$$\{b_r^\mu, b_s^\nu\}_{D.B.} = -i\eta^{\mu\nu} \delta_{r+s}, \quad \{\bar{b}_r^\mu, \bar{b}_s^\nu\}_{D.B.} = -i\eta^{\mu\nu} \delta_{r+s}, \quad \{\bar{b}_r^\mu, b_s^\nu\}_{D.B.} = 0 \tag{2.40}$$

It is now possible to compute the classic super Virasoro algebra of the constraints in the (NS,NS) sector[9]

$$\{\mathcal{L}_n, \mathcal{L}_m\}_{D.B.} = -i(n-m)\mathcal{L}_{n+m}, \quad \{\mathcal{L}_n, \mathcal{Q}_r\}_{D.B.} = -i\left(\frac{n}{2} - r\right) \mathcal{Q}_{n+r} \tag{2.41}$$

$$\{\mathcal{Q}_r, \mathcal{Q}_s\}_{D.B.} = -2i\mathcal{L}_{r+s}$$

$$\{\bar{\mathcal{L}}_n, \bar{\mathcal{L}}_m\}_{D.B.} = -i(n-m)\bar{\mathcal{L}}_{n+m}, \quad \{\bar{\mathcal{L}}_n, \bar{\mathcal{Q}}_r\}_{D.B.} = -i\left(\frac{n}{2} - r\right) \bar{\mathcal{Q}}_{n+r} \tag{2.42}$$

$$\{\bar{\mathcal{Q}}_r, \bar{\mathcal{Q}}_s\}_{D.B.} = -2i\bar{\mathcal{L}}_{r+s}$$

For the present work, we have only shown the necessary results for the description of the (NS,NS) sector. this thesis project.

2.2 Quantum Description

This section explores the covariant quantization of superstring theory in the RNS formalism, a method that maintains both Lorentz invariance and worldsheet supersymmetry while accommodating the fermionic and bosonic degrees of freedom fundamental to superstring dynamics. The section begins by applying the canonical quantization to the boson and fermionic fields. The subsequent analysis is divided into open strings and closed strings, where distinct boundary conditions lead to different classes of states. The quantization process ends by imposing the constraints from the quantum super-Virasoro algebra, essential for enforcing physical state conditions.

2.2.1 Covariant Quantization

We proceed in the same way as in the bosonic string. That is, we treat all string fields $X^\mu(\tau, \sigma)$ and $\psi^\mu(\tau, \sigma)$ on the same footing to maintain the Lorentz invariance. We make the transition from classical mechanics to quantum mechanics replacing the classical brackets by the commutator and anti-commutator:

$$\{ \ , \ }_{PB} \rightarrow \frac{1}{i} [\ , \] \quad \{ \ , \ }_{DB} \rightarrow \frac{1}{i} \{ \ , \ }. \quad (2.43)$$

$$\begin{aligned} [X^\mu(\sigma, \tau), \mathcal{P}^\nu(\sigma', \tau)] &= i\eta^{\mu\nu} \delta(\sigma - \sigma'), \\ \{\psi_+^\mu(\tau, \sigma), \psi_+^\nu(\tau, \sigma')\} &= 2\pi\alpha' \eta^{\mu\nu} \delta(\sigma - \sigma'), \\ \{\psi_-^\mu(\tau, \sigma), \psi_-^\nu(\tau, \sigma')\} &= 2\pi\alpha' \eta^{\mu\nu} \delta(\sigma - \sigma'), \\ \{\psi_+^\mu(\tau, \sigma), \psi_-^\nu(\tau, \sigma')\} &= 0 \end{aligned} \quad (2.44)$$

From (2.44), the commutation and anti-commutation relations for the oscillator modes in the NS sector are obtained. For the different string configurations, the resulting expressions take the form of

Closed string (NS,NS)

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}, \quad n, m \in \mathbb{Z} \\ \{b_r^\mu, b_s^\nu\} &= \{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \delta_{r+s}\eta^{\mu\nu}, \quad r, s \in \mathbb{Z} + 1/2. \end{aligned} \quad (2.45)$$

Open string (NS)

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= m\delta_{m+n}\eta^{\mu\nu}, \quad n, m \in \mathbb{Z} \\ \{b_r^\mu, b_s^\nu\} &= \delta_{r+s}\eta^{\mu\nu}, \quad r, s \in \mathbb{Z} + 1/2. \end{aligned} \quad (2.46)$$

In addition, the reality condition on X^μ and ψ^μ implies that $\alpha_m^{\mu\dagger} = \alpha_{-m}^\mu$ and $b_r^{\mu\dagger} = b_{-r}^\mu$. We again see that we can split oscillators into creation and annihilation operators according to the sign of their n, r index. Oscillators with $n, r > 0$ are annihilation operators, and oscillators with $n, r < 0$ are creation operators.

With the commutation and the anti-commutation rules (2.45), (2.46), the oscillator expansions in (2.35), (2.37) can be normal ordered in the quantum regime. As in the bosonic string case, the only generator that has a normal ambiguity is $\mathcal{L}_0$, so there exist a normal ordering constants for this case. The quantum oscillator expansions or super Virasoro constraints $\mathcal{L}_n, \bar{\mathcal{L}}_n, \mathcal{Q}_r, \bar{\mathcal{Q}}_r$ are the generators of the super Virasoro algebra, and satisfy the following relations in the NS sector [11]

Closed string (NS,NS)

$$\begin{aligned} [\mathcal{L}_n, \mathcal{L}_m] &= (n-m)\mathcal{L}_{m+n} + \frac{D}{8}\delta_{m+n}(n^3-n), \\ [\bar{\mathcal{L}}_n, \bar{\mathcal{L}}_m] &= (n-m)\bar{\mathcal{L}}_{m+n} + \frac{D}{8}\delta_{m+n}(n^3-n), \\ [\mathcal{L}_n, \mathcal{Q}_r] &= \left(\frac{n}{2} - r\right) \mathcal{Q}_{n+r}, \\ [\bar{\mathcal{L}}_n, \bar{\mathcal{Q}}_r] &= \left(\frac{n}{2} - r\right) \bar{\mathcal{Q}}_{n+r}, \\ \{\mathcal{Q}_r, \mathcal{Q}_s\} &= 2\mathcal{L}_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}, \\ \{\bar{\mathcal{Q}}_r, \bar{\mathcal{Q}}_s\} &= 2\bar{\mathcal{L}}_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s} \end{aligned} \quad (2.47)$$

Open string (NS)

$$\begin{aligned} [\mathcal{L}_n, \mathcal{L}_m] &= (n-m)\mathcal{L}_{m+n} + \frac{D}{8}\delta_{m+n}(n^3-n), \\ [\mathcal{L}_n, \mathcal{Q}_r] &= \left(\frac{n}{2} - r\right) \mathcal{Q}_{n+r}, \\ \{\mathcal{Q}_r, \mathcal{Q}_s\} &= 2\mathcal{L}_{r+s} + \frac{D}{2} \left(r^2 - \frac{1}{4}\right) \delta_{r+s} \end{aligned} \quad (2.48)$$

Let us now examine the states in the Hilbert space of the theory in the NS sector. The oscillator ground state or vacuum in this sector is defined by

Closed string (NS,NS)

$$\alpha_m^\mu |0, k^\mu\rangle = \bar{\alpha}_m^\mu |0, k^\mu\rangle = b_r^\mu |0, k^\mu\rangle = \bar{b}_r^\mu |0, k^\mu\rangle = 0, \quad m = 1, 2, \dots, \quad r = \frac{1}{2}, \frac{3}{2}, \dots \quad (2.49)$$

Open string (NS)

$$\alpha_m^\mu |0, k^\mu\rangle = b_r^\mu |0, k^\mu\rangle = 0, \quad m = 1, 2, \dots, \quad r = \frac{1}{2}, \frac{3}{2}, \dots \quad (2.50)$$

The general state basis $|\Lambda\rangle$ in this Hilbert space is built up of oscillators acting on the vacuum [26].

Closed String (NS,NS)

$$|\Lambda\rangle = \left[\prod_{n=1}^{\infty} \prod_{\mu=0}^D (\alpha_n^{\mu\dagger})^{\lambda_{n,\mu}} \right] \left[\prod_{m=1}^{\infty} \prod_{\nu=0}^D (\bar{\alpha}_m^{\nu\dagger})^{\bar{\lambda}_{m,\nu}} \right] \left[\prod_{r=1/2}^{\infty} \prod_{\alpha=0}^D (b_r^{\alpha\dagger})^{\Lambda_{r,\alpha}} \right] \left[\prod_{s=1/2}^{\infty} \prod_{\beta=0}^D (\bar{b}_s^{\beta\dagger})^{\bar{\Lambda}_{s,\beta}} \right] |0, k\rangle, \quad (2.51)$$

Open String (NS)

$$|\Lambda\rangle = \left[\prod_{n=1}^{\infty} \prod_{\mu=0}^D (\alpha_n^{\mu\dagger})^{\lambda_{n,\mu}} \right] \left[\prod_{r=1/2}^{\infty} \prod_{\alpha=0}^D (b_r^{\alpha\dagger})^{\Lambda_{r,\alpha}} \right] |0, k\rangle. \quad (2.52)$$

The next step in the quantization procedure is to give the physical state conditions. As in the bosonic case, we impose only half of the constraints on physical states $|\Phi\rangle$ [24].

Closed string (NS,NS)

$$\begin{aligned} \mathcal{Q}_r |\Phi\rangle &= 0, \quad \bar{\mathcal{Q}}_r |\Phi\rangle = 0 \quad r \geq 1/2, \\ \mathcal{L}_n |\Phi\rangle &= 0, \quad \bar{\mathcal{L}}_n |\Phi\rangle = 0 \quad n \geq 1, \\ (\mathcal{L}_0 - a) |\Phi\rangle &= 0, \quad (\bar{\mathcal{L}}_0 - \bar{a}) |\Phi\rangle = 0 \end{aligned} \quad (2.53)$$

Open string (NS)

$$\begin{aligned} \mathcal{Q}_r |\Phi\rangle &= 0, \quad r \geq 1/2, \\ \mathcal{L}_n |\Phi\rangle &= 0, \quad n \geq 1, \\ (\mathcal{L}_0 - a) |\Phi\rangle &= 0, \quad (\bar{\mathcal{L}}_0 - \bar{a}) |\Phi\rangle = 0 \end{aligned} \quad (2.54)$$

where the constants $a, \bar{a}$ are required to take into account the ambiguity ordering in the zero Virasoro constraint.

In the case of the open string the (NS) sector contains bosonic states, and the Ramond (R) sector contains fermionic states. For the closed string, the (NS,NS) and (R,R) sectors give bosons states, while the (R,NS) and (NS,R) sectors give fermions states [8]. These physical states are consistent in a ten dimensional space-time. For the space-time dimension $D \leq 10$ the states $|\Phi\rangle$ which satisfy the relations (2.53) and (2.54) have positive norm (no-ghost theorem).

Chapter 3

The Tensionless Bosonic String

In chapter 1 we studied the bosonic string with tension $T \neq 0$. In this chapter, we will explore the zero tension limit, in the classical and quantum regime, of the bosonic string theory. We closely follow [15, 2, 5, 3, 1]. We start from the Hamiltonian with Lagrange multipliers to find the bosonic tensionless string action, and we will describe its symmetries. From the tensionless action, the classical equations of restriction are obtained, which are promoted to operators in the quantum description to filter out the physical states. Then, we will define the vacuum state for the tensionless closed string. Finally, through a Bogoliubov transformation, we link the tensionless vacuum to the usual tensile vacuum, and we will see that it behaves as an open string vacuum in the ultra relativistic limit.

3.1 Classical Description

3.1.1 Tensionless Action

We want to find an action in the tensionless limit. If we try to take the limit $T \rightarrow 0$ in the Nambu-Goto action, because of the square root, derivative terms, and the tensile dependence in the string coordinates $X^\mu(\tau, \sigma; T)$, it would lead to very long calculations. A better approach to find such tensionless action is to propose an equivalent action in which the tension T is explicitly written in a different way.

Let's consider the formalism for obtaining the equations of motion from the extended Hamilton's action principle. That is, for a system with m constrain equations $\phi_m(q, p) = 0$, the action for such system can be written as [13]

$$S = \int dt L = \int dt [\dot{q}^n p_n - H_c - u^m \phi_m], \quad (3.1)$$

where $u^m(q, p)$ are Lagrange multipliers, and the equations of motion are given through arbitrary variations $\delta q^n, \delta p_n, \delta u_m$ of the action.

$$\dot{q}^n = \frac{\partial H_c}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}, \quad \dot{p}_n = -\frac{\partial H_c}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n}, \quad \phi_m = 0. \quad (3.2)$$

In this way, the constrain equations of the Nambu-Goto action can be used to find an equivalent action. It is known that there are only two equations of restriction for

this action (1.14):

$$\begin{aligned}\phi_1 &= \mathcal{P}^2 + T^2(X')^2 = 0, \\ \phi_2 &= \mathcal{P} \cdot X' = 0.\end{aligned}\tag{3.3}$$

In addition, the canonical Hamiltonian is such that

$$\mathcal{H}_c = \dot{X}^\mu \mathcal{P}_\mu - \mathcal{L}_{NG} = -T \frac{(\sqrt{-\gamma})^2}{\sqrt{-\gamma}} + T\sqrt{-\gamma} = 0,\tag{3.4}$$

as expected for a diffeomorphism invariant theory.

On the other hand, the momentum $\mathcal{P}^\mu$ can be written in terms of $\dot{X}^\mu$ using one of the relations in (3.2). That is,

$$\dot{X}^\mu = \frac{\partial \mathcal{H}_c}{\partial \mathcal{P}_\mu} + u^m \frac{\partial \phi_m}{\partial \mathcal{P}_\mu} = \lambda \frac{\partial \phi_1}{\partial \mathcal{P}_\mu} + \rho \frac{\partial \phi_2}{\partial \mathcal{P}_\mu} = 2\lambda \mathcal{P}^\mu + \rho X'^\mu\tag{3.5}$$

$$\Rightarrow \mathcal{P}^\mu = \frac{1}{2\lambda}(\dot{X}^\mu - \rho X'^\mu)\tag{3.6}$$

Then, following [15] we rewrite the Nambu-Goto action in terms of $\dot{X}^\mu, X'^\mu$:

$$\begin{aligned}S &= \int d\tau d\sigma \left(\dot{X}^\mu \mathcal{P}_\mu - \mathcal{H}_c - \lambda \phi_1 - \rho \phi_2 \right), \\ &= \frac{1}{2} \int d\tau d\sigma \frac{1}{2\lambda} \left[(\dot{X})^2 + \rho^2 (X')^2 - 2\rho \dot{X} \cdot X' - 4\lambda^2 T^2 (X')^2 \right], \\ &= -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \\ &= S[X^\mu, h^{\alpha\beta}].\end{aligned}\tag{3.7}$$

The previous action shows that if $h^{\alpha\beta}$ is defined as (3.8), then S can be identified with the Polyakov action [3]. This is an important result since now we can use all the symmetries related to this known action to fix the values of the Lagrange multipliers λ and ρ . The relationship between the worldsheet metric tensor $h^{\alpha\beta}$ and the Lagrange multipliers is given by:

$$h^{\alpha\beta} = \begin{pmatrix} -1 & \rho \\ \rho & -\rho^2 + 4\lambda^2 T^2 \end{pmatrix}, \quad h = \det(h_{\alpha\beta}) = -\frac{1}{4\lambda^2 T^2}.\tag{3.8}$$

In the tensionless limit $T \rightarrow 0$, the action (3.7) and $T\sqrt{-h}h^{\alpha\beta}$ take the following form

$$S = \frac{1}{2} \int d\tau d\sigma \frac{1}{2\lambda} \left[(\dot{X})^2 + \rho^2 (X')^2 - 2\rho \dot{X} \cdot X' \right],\tag{3.9}$$

$$T\sqrt{-h}h^{\alpha\beta} = \frac{1}{2\lambda} \begin{pmatrix} -1 & \rho \\ \rho & -\rho^2 \end{pmatrix}.\tag{3.10}$$

The previous expressions motivate the introduction of a Lorentz vector $V^\alpha = 1/\sqrt{2}\lambda(1, \rho)$ to rewrite the bosonic tensionless action (3.9) as in [15]

$$S[X^\mu, V^\alpha] = \int d\tau d\sigma V^\alpha V^\beta \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}.\tag{3.11}$$

This new action is invariant under Poincare and diffeomorphism transformations[15]. Here, we will show explicitly the second symmetry. In order to maintain the diffeomorphism invariance, V^α must behave as a vector density (weighted tensor) under a reparametrization. That is,

$$\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha(\sigma^\beta), \quad \tilde{V}^\alpha = \left[\det \left(\frac{\partial \tilde{\sigma}^a}{\partial \sigma^b} \right) \right]^{-1/2} \frac{\partial \sigma^\alpha}{\partial \tilde{\sigma}^\beta} V^\beta(\tilde{\sigma}^c) \quad (3.12)$$

$$\begin{aligned} \tilde{V}^\alpha \tilde{V}^\beta &= J^{-1/2} \frac{\partial \sigma^\alpha}{\partial \tilde{\sigma}^a} V^a(\tilde{\sigma}) J^{-1/2} \frac{\partial \sigma^\beta}{\partial \tilde{\sigma}^b} V^b(\tilde{\sigma}), \quad J = \det \left(\frac{\partial \sigma^a}{\partial \tilde{\sigma}^b} \right) \\ &= J^{-1} V^a(\tilde{\sigma}) V^b(\tilde{\sigma}) \frac{\partial \sigma^\beta}{\partial \tilde{\sigma}^b} \frac{\partial \sigma^\alpha}{\partial \tilde{\sigma}^a} \end{aligned} \quad (3.13)$$

$$\begin{aligned} S &= \int d\tau d\sigma \tilde{V}^\alpha(\tilde{\sigma}) \tilde{V}^\beta(\tilde{\sigma}) \partial_\alpha X^\mu(\tilde{\sigma}) \partial_\beta X^\nu(\tilde{\sigma}) \eta_{\mu\nu}, \\ &= \int d\tau d\sigma J^{-1} V^a(\tilde{\sigma}) V^b(\tilde{\sigma}) \frac{\partial \sigma^\beta}{\partial \tilde{\sigma}^b} \frac{\partial \sigma^\alpha}{\partial \tilde{\sigma}^a} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \\ &= \int d\tilde{\tau} d\tilde{\sigma} V^{\tilde{\alpha}}(\tilde{\sigma}) V^{\tilde{\beta}}(\tilde{\sigma}) \partial_{\tilde{\alpha}} X^\mu \partial_{\tilde{\beta}} X^\nu \eta_{\mu\nu}, \\ &= \tilde{S} \end{aligned} \quad (3.14)$$

then, the action is invariant under diffeomorphism transformations.

3.1.2 Equations of Motion and Mode Expansions

The analysis of bosonic tensionless string theory is extended in [3] following [15]. Specifically, the analysis of the equations of motion, its mode expansion and constraints are covered. We show here some of the results from [3] with more detail. Let's start with the variation of the action $\delta S = 0$ defined in (3.11) to find the equations of motion.

$$\begin{aligned} &\int d\tau d\sigma 2\delta V^\alpha (V^\beta \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}) + 2V^\alpha V^\beta \delta(\partial_\alpha X^\mu) \partial_\beta X^\nu \eta_{\mu\nu} = 0, \\ &= \int d\tau d\sigma \delta V^\alpha (V^\beta \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}) - \delta X^\mu \partial_\alpha (V^\alpha V^\beta \partial_\beta X^\nu \eta_{\mu\nu}), \end{aligned} \quad (3.15)$$

$$\Rightarrow \partial_\alpha (V^\alpha V^\beta \partial_\beta X^\mu) = 0, \quad V^\beta \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} = 0, \quad (3.16)$$

where integration by parts and periodic boundary conditions were applied in (3.15). In order to obtain simpler motion equations, symmetries must be taken into account. In the conformal gauge, $h^{\alpha\beta} = \eta^{\alpha\beta}$ and so $\rho = 0$ in (3.8). In addition, the tensionless action is invariant under diffeomorphism, so it's possible to take the fixed gauge $V^\alpha = (v, 0)$ where v is a constant [15]. This gauge corresponds to the conformal gauge in the tensile theory. Then, using this gauge the motion equations (3.16) are such that

$$\ddot{X}^\mu = 0, \quad \dot{X}^2 = 0, \quad \dot{X} \cdot X' = 0. \quad (3.17)$$

For a particular Lorentz observer $X^0 = \tau$, then the above equations imply the following expressions:

$$\ddot{X}^\mu = 0, \quad \dot{X}^2 = 1, \quad \dot{X} \cdot \dot{X}' = 0. \quad (3.18)$$

Therefore, the motion of the string is restricted:

1. Every point of the string moves with speed of light.
2. Longitudinal oscillations are forbidden.

The solution for $\ddot{X}^\mu = 0$ can be calculated by noting that for a closed string the periodicity condition $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$ implies the existence of a Fourier series such that

$$X^\mu(\tau, \sigma) = x^\mu + \tilde{A}_0^\mu \sigma + \sum_{n \in \mathbb{Z}} C_n^\mu(\tau) e^{-in\sigma}. \quad (3.19)$$

Since $\ddot{X}^\mu = 0$, then $C_n^\mu = \tilde{A}_n^\mu + \tilde{B}_n^\mu \tau$. Furthermore, $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$ implies $\tilde{A}_0^\mu = 0$. It's convenient to choose $\tilde{A}_n^\mu = i\sqrt{2c'} A_n^\mu / n$ and $\tilde{B}_n^\mu = \sqrt{2c'} n B_n^\mu / n$ for $n \neq 0$ and $C_0^\mu = \sqrt{2c'} B_0^\mu$. Then, [3]

$$X^\mu(\tau, \sigma) = x^\mu + \sqrt{2c'} B_0^\mu \tau + i\sqrt{2c'} \sum_{n \neq 0} \frac{1}{n} (A_n^\mu - in\tau B_n^\mu) e^{-in\sigma}. \quad (3.20)$$

The constrain equations in (3.17) can be written in terms of oscillator modes, then they translate to:

$$\begin{aligned} \dot{X}^2 &= 2c' \sum_{m,n} B_n \cdot B_m e^{-i\sigma(m+n)} = 2c' \sum_{m,n} B_{-n} \cdot B_{-m} e^{i\sigma(m+n)}, \\ &= 2c' \sum_{m,k} B_{-m} \cdot B_{m+k} e^{-ik\sigma} = 4c' \sum_{n \in \mathbb{Z}} M_n e^{-in\sigma} = 0, \end{aligned} \quad (3.21)$$

$$\begin{aligned} X' \cdot \dot{X} &= 2c' \sum_{n,m} (A_n - in\tau B_n) \cdot B_m e^{-i\sigma(n+m)} = 2c' \sum_{n,m} (A_{-m} - in\tau B_{-m}) \cdot B_{m+n} e^{-in\sigma}, \\ &= 4c' \sum_{n \in \mathbb{Z}} (L_n - in\tau M_n) e^{-in\sigma} = 0, \end{aligned} \quad (3.22)$$

where we have defined L_n and M_n as

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} A_{-m} \cdot B_{m+n}, \quad M_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} B_{-m} \cdot B_{m+n}. \quad (3.23)$$

Therefore, from (3.21) and (3.22), we arrive to the constrain equations [3]

$$M_n = 0, \quad L_n = 0, \quad \forall n \in \mathbb{Z}. \quad (3.24)$$

3.1.3 Generators

A residual symmetry that leaves the gauge fixed action invariant remains. In order to find it, let's consider the infinitesimal transformation of diffeomorphism $\sigma^\alpha \rightarrow \sigma^\alpha + \epsilon^\alpha$ in (3.12):

$$\begin{aligned}\tilde{V}^\alpha &= (1 + \partial_\alpha \epsilon^\alpha)^{1/2} (\delta_\beta^\alpha + \partial_\beta \epsilon^\alpha) (V^\beta + \epsilon^a \partial_a V^\beta), \\ &= V^\alpha + V^\beta \partial_\beta \epsilon^\alpha + \epsilon^\beta \partial_\beta V^\alpha + \frac{1}{2} V^\alpha \partial_\beta \epsilon^\beta.\end{aligned}\quad (3.25)$$

By requiring invariance, $\delta V^\alpha = \tilde{V}^\alpha - V^\alpha = 0$, and using the gauge $V^\alpha = (v, 0)$ then

$$\begin{aligned}-\dot{\epsilon}^0 + \frac{1}{2}\dot{\epsilon}^0 + \frac{1}{2}(\epsilon^1)' &= 0, \\ -v\dot{\epsilon}^1 &= 0.\end{aligned}\quad (3.26)$$

The previous system of equations is only satisfied if $\epsilon^\alpha = (f'(\sigma)\tau + g(\sigma), g(\sigma))$, where $f(\sigma), g(\sigma)$ are arbitrary well behaved functions. Now, it's possible to see the effect of such transformations on coordinates (or any function):

$$\begin{aligned}X^\mu(\tau + \epsilon^0, \sigma + \epsilon^1) &= X^\mu(\tau, \sigma) + \epsilon^\alpha \partial_\alpha X^\mu(\tau, \sigma), \\ &= [1 + L(f) + M(g)] X^\mu(\sigma, \tau),\end{aligned}\quad (3.27)$$

where $L(f) = f'(\sigma)\tau\partial_\tau + f(\sigma)\partial_\sigma$ and $M(g) = g(\sigma)\partial_\tau$. These operators satisfy the following commutation relations[15]:

$$\begin{aligned}[L(f_1), L(f_2)] &= L(f_1 f_2' - f_1' f_2), \\ [L(f), M(g)] &= M(fg' - f'g), \\ [M(g_1), M(g_2)] &= 0.\end{aligned}\quad (3.28)$$

In terms of Fourier modes, $f(\sigma) = \sum a_n e^{in\sigma}$, $g(\sigma) = \sum b_n e^{in\sigma}$, then it holds that

$$\begin{aligned}L(f) &= \sum a_n e^{in\sigma} (\partial_\sigma + in\tau\partial_\tau) = -i \sum a_n L_n, & L_n &= ie^{in\sigma} (\partial_\sigma + in\tau\partial_\tau) \\ M(g) &= \sum b_n e^{in\sigma} \partial_\tau = -i \sum b_n M_n, & M_n &= ie^{in\sigma} \partial_\tau\end{aligned}\quad (3.29)$$

Substituting the above relations into the commutation relations (3.28) the following commutation expressions are obtained:

$$\begin{aligned}[L_m, L_n] &= (m - n)L_{m+n}, \\ [L_m, M_n] &= (m - n)M_{m+n}, \\ [M_m, M_n] &= 0.\end{aligned}\quad (3.30)$$

This algebra and the generators can also be derived from a contraction of the classical Virasoro algebra where the constants $c, \bar{c}$ (central charges) are zero for the closed string. To prove the above statement, let's consider the classical Virasoro generators $\mathcal{L}_n, \bar{\mathcal{L}}_m$ in the following representation [1]

$$\mathcal{L}_n = ie^{in\omega} \partial_\omega, \quad \bar{\mathcal{L}}_n = ie^{in\bar{\omega}} \partial_{\bar{\omega}}, \quad (3.31)$$

where $\omega = \tau + \sigma$ and $\bar{\omega} = \tau - \sigma$. Then,

$$\begin{aligned}\mathcal{L}_n - \bar{\mathcal{L}}_{-n} &= ie^{in\sigma} [i \sin(n\tau) \partial_\tau + \cos(n\tau) \partial_\sigma], \\ \mathcal{L}_n + \bar{\mathcal{L}}_{-n} &= ie^{in\sigma} [\cos(n\tau) \partial_\tau + i \sin(n\tau) \partial_\sigma]\end{aligned}\quad (3.32)$$

If we perform the Carrollian or ultrarelativistic limit $c = \epsilon \rightarrow 0$ then $\tau \rightarrow \epsilon\tau$, $\sigma \rightarrow \sigma$ in $\mathcal{L}_n - \bar{\mathcal{L}}_{-n}$ and $\epsilon(\mathcal{L}_n + \bar{\mathcal{L}}_{-n})$, then

$$\begin{aligned}\mathcal{L}_n - \bar{\mathcal{L}}_{-n} &= ie^{in\sigma} \left(i\epsilon n\tau \frac{\partial}{\epsilon \partial \tau} + \partial_\sigma \right) = L_n, \\ \epsilon(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}) &= ie^{in\sigma} \left(\epsilon \frac{\partial}{\epsilon \partial \tau} + \epsilon^2 \tau n \partial_\sigma \right), \\ &= ie^{in\sigma} \partial_\tau = M_n.\end{aligned}\quad (3.33)$$

This result suggests that the tensile theory in the ultrarelativistic limit could be related to the tensionless theory .

3.1.4 Limit from the Tensile String

In order to relate the tensile, in the aforementioned limit, and tensionless theory, we follow [3]. We consider the equation of motion from the tensile theory (1.38) and the constrain equations (1.40) in the ultra-relativistic limit $c = \epsilon \rightarrow 0$:

$$\ddot{X}^\mu - \epsilon^2 X''^\mu = 0, \quad \dot{X} \cdot X' = 0, \quad \ddot{X}^2 + \epsilon^2 X'^2 = 0, \quad (3.34)$$

In addition, the solution to the wave equation for the closed string (1.43) in the ultra-relativistic limit is such that

$$\begin{aligned}X^\mu(\tau, \sigma) &= x^\mu + 2\sqrt{2c'}\sqrt{\epsilon}\alpha_0^\mu\tau \\ &+ i\sqrt{2c'}\sum_{n \neq 0} \frac{1}{n} \left[\frac{\alpha_n^\mu - \bar{\alpha}_{-n}^\mu}{\sqrt{\epsilon}} - in\tau\sqrt{\epsilon}(\alpha_n^\mu + \bar{\alpha}_{-n}^\mu) \right] e^{-in\sigma}\end{aligned}\quad (3.35)$$

where $c' = 1/2\pi T$. As can be seen, the solution in this limit is of order $O(\epsilon^{-1/2})$, then $\epsilon^2 X''^\mu$ and $\epsilon^2 X'^2$ vanish in the limit (3.34) . Therefore,

$$\ddot{X}^\mu = 0, \quad \dot{X}^2 = 0, \quad \dot{X} \cdot X' = 0. \quad (3.36)$$

The equations (3.36) reproduce the previous result from tensionless theory (3.17). Thus, the solutions (3.20) and (3.35) must coincide. A comparison between these solutions reveals that [3]

$$A_n^\mu = \frac{1}{\sqrt{\epsilon}}(\alpha_n^\mu - \bar{\alpha}_{-n}^\mu), \quad B_n^\mu = \sqrt{\epsilon}(\alpha_n^\mu + \bar{\alpha}_{-n}^\mu). \quad (3.37)$$

In this way, the algebra of A_n^μ, B_n^μ can be determined using the Poisson brackets relations of the tensile closed string (1.48):

$$\begin{aligned}\{A_m^\mu, B_n^\nu\} &= \left\{ \frac{1}{\sqrt{\epsilon}}(\alpha_m^\mu - \bar{\alpha}_{-m}^\mu), \sqrt{\epsilon}(\alpha_n^\nu + \bar{\alpha}_{-n}^\nu) \right\}, \\ &= \{\alpha_m^\mu, \alpha_n^\nu\} - \{\bar{\alpha}_{-m}^\mu, \bar{\alpha}_{-n}^\nu\}, \\ &= -2im\delta_{m+n,0}\eta^{\mu\nu}\end{aligned}\quad (3.38)$$

$$\{A_m^\mu, A_n^\nu\} = \{B_m^\mu, B_n^\nu\} = 0 \quad (3.39)$$

As can be seen in (3.38) and (3.39) these one are not the algebra of harmonic oscillator modes (1.48).

3.2 Quantum Description

3.2.1 2d Galilean Conformal Algebra

If we promote the classical tensionless theory of closed string through canonical quantization $\{ \quad \} \rightarrow -i[\quad]$, it is necessary to consider nonzero central charges $c, \bar{c}$ as a consequence of non commutativity of some operators $\alpha_n, \bar{\alpha}_n$ in Virasoro generators. Thus, we consider $\mathcal{L}_m, \bar{\mathcal{L}}_n$ that satisfy the following relations:

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\ [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] &= (m-n)\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}, \\ [\mathcal{L}_m, \bar{\mathcal{L}}_n] &= 0. \end{aligned} \quad (3.40)$$

These Virasoro operators replace the classical operators in the expressions (3.33) to extend the algebra (3.30) [1] :

$$L_n = \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \quad M_n = \epsilon(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}) \quad (3.41)$$

$$\begin{aligned} [L_m, M_n] &= [\mathcal{L}_m - \bar{\mathcal{L}}_{-m}, \epsilon(\mathcal{L}_n + \bar{\mathcal{L}}_{-n})] \\ &= \epsilon \left((m-n)\mathcal{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} - (-m+n)\bar{\mathcal{L}}_{-m-n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{-m-n,0} \right) \\ &= \epsilon(m-n)(\mathcal{L}_{m+n} + \bar{\mathcal{L}}_{-m-n}) + \epsilon \frac{c+\bar{c}}{12}m(m^2-1)\delta_{m+n,0} \\ &= M_{m+n} + \frac{c_M}{12}m(m^2-1)\delta_{m+n,0}. \end{aligned} \quad (3.42)$$

Other combinations of commutators are obtained in a similar way. The generators form the 3d Bondi-Metzner-Sachs algebra (BMS_3) or equivalently the 2d Galilean Conformal Algebra (2d-GCA) [1]

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c_L}{12}m(m^2-1)\delta_{m+n,0}, \\ [L_m, M_n] &= (m-n)M_{m+n} + \frac{c_M}{12}m(m^2-1)\delta_{m+n,0}, \\ [M_m, M_n] &= 0, \end{aligned} \quad (3.43)$$

where $c_M = \epsilon(c+\bar{c})$ and $c_L = c-\bar{c}$. For a well defined string theory $c = \bar{c} = D$, then c_L, c_M vanish in the ultra relativistic limit . If the central charges are not zero, it is possible to think of tensionless strings which are not derived as a limit from tensile string theories and are fundamental objects in their own right [3].

3.2.2 Quantum Constrains

The quantum version of constraints (3.24) allows to restrict the Hilbert space to filter out the physical states. The most general way to impose this condition is demanding all the matrix elements of the constraint acting on physical states to vanish.

$$\langle phys | L_n | phys \rangle = 0 = \langle phys | M_n | phys \rangle, \quad \forall n \in \mathbb{Z}. \quad (3.44)$$

There are several ways to impose the constrains. Specifically, they can be splitted out in three different cases^[5]:

$$\begin{aligned} F_n | phys \rangle &= 0, & n > 0 \\ F_n | phys \rangle &= 0, & n \neq 0 \\ F_n | phys \rangle &\neq 0 & \forall n \in \mathbb{Z}. \end{aligned} \quad (3.45)$$

Since the zero components of A_m^0, B_n^0 could have ordering ambiguities, additional constants must be considered: $F_0 =: F_0 : -a_F$. In order to impose the physical conditions, the vacuum state is considered as a physical state with respect to the constraints. According to the relations (3.45), there are nine possible ways to obtain vacuums $|0, k^\mu\rangle$, and restrict the Hilbert space. However, in [5], it is proved that only four vacuums are physically consistent. The main three cases are given by ¹

$$L_m |0, k^\mu\rangle = 0, \quad M_n |0, k^\mu\rangle = 0, \quad m > 0, n > 0 \quad (3.46)$$

$$L_m |0, k^\mu\rangle \neq 0, \quad M_n |0, k^\mu\rangle = 0, \quad m \in \mathbb{Z}, n \neq 0 \quad (3.47)$$

$$L_m |0, k^\mu\rangle \neq 0, \quad M_n |0, k^\mu\rangle \neq 0, \quad m, n \in \mathbb{Z}. \quad (3.48)$$

Because we are interested in the evolution of the tensile theory under the tensionless limit, only one of the physical vacuum is considered: the induced vacuum. This is given by the relation (3.47) and we denote it as

$$|0, k^\mu\rangle = |0, k^\mu\rangle_I, \quad (3.49)$$

Therefore, for $m, n \in \mathbb{Z}$

$$\begin{aligned} (L_m - a_L \delta_{m,0}) |0, k^\mu\rangle_I &= \sum_{s \in \mathbb{Z}} A_{-s} \cdot B_{s+m} |0, k^\mu\rangle_I \neq 0, \\ (M_n - a_M \delta_{n,0}) |0, k^\mu\rangle_I &= \sum_{s \in \mathbb{Z}} B_{-s} \cdot B_{s+n} |0, k^\mu\rangle_I = 0. \end{aligned} \quad (3.50)$$

Here, the constants a_M, a_L have been added to consider ordering ambiguities in the zero component of the operator M_n, L_m . In terms of oscillator modes it holds that

$$A_m^\mu |0, k^\mu\rangle_I \neq 0, \quad B_n^\mu |0, k^\mu\rangle_I = 0, \quad B_0^\mu |0, k^\mu\rangle_I = b_0 |0, k^\mu\rangle_I. \quad (3.51)$$

¹The remaining case occurs when $k^\mu = 0$ for all n, m .

3.2.3 Induced Representation

According to [2], the induced vacuum (3.51) can be settled in a subalgebra of BMS_3 . Following this work, let's consider $|h, \bar{h}\rangle$ as the physical states for the closed string in the tensile theory. Then, the Virasoro operators over these states satisfy the following relations:

$$\mathcal{L}_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle, \quad \bar{\mathcal{L}}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle, \quad (3.52)$$

$$\mathcal{L}_n |h, \bar{h}\rangle = 0 = \bar{\mathcal{L}}_n |h, \bar{h}\rangle, \quad \forall n > 0. \quad (3.53)$$

In the ultrarelativistic limit, the Virasoro operators can be expressed in terms of the operators of the Galilean Conformal Algebra 2D through inverting the relation (3.41). So that,

$$\begin{aligned} \mathcal{L}_0 |h, \bar{h}\rangle &= \frac{1}{2} \left(L_0 + \frac{M_0}{\epsilon} \right) |h, \bar{h}\rangle = h |h, \bar{h}\rangle, \\ \bar{\mathcal{L}}_0 |h, \bar{h}\rangle &= \frac{1}{2} \left(\frac{M_0}{\epsilon} - L_0 \right) |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle. \end{aligned} \quad (3.54)$$

If the expressions (3.54) are subtracted and added, then

$$\begin{aligned} M_0 |h, \bar{h}\rangle &= \epsilon(h + \bar{h}) |h, \bar{h}\rangle, \\ L_0 |h, \bar{h}\rangle &= (h - \bar{h}) |h, \bar{h}\rangle, \\ M_n |h, \bar{h}\rangle &= \epsilon(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}) |h, \bar{h}\rangle = 0. \end{aligned} \quad (3.55)$$

If we assume that in the ultrarelativistic limit, the Virasoro operators map as $|h, \bar{h}\rangle = |M, s\rangle$ when $\epsilon \rightarrow 0$, the representations become induced representations of BMS_3 in this limit [2].

$$M_0 |M, s\rangle = M |M, s\rangle, \quad L_0 |M, s\rangle = s |M, s\rangle, \quad M_n |M, s\rangle = 0 \quad n \neq 0 \quad (3.56)$$

where $M = \epsilon(h + \bar{h})$ and $s = h - \bar{h}$. This defines a 1D representation of the subalgebra of BMS_3 spanned by $\{L_0, M_n, c_L; c_M\}$. For $|M, s\rangle = |0, k^\mu\rangle_I$, the induced vacuum is recovered in this representation as can be recognized in (3.50) or (3.51).

3.2.4 Mass Spectrum

Other aspects of bosonic tensionless strings such as the mass have been studied in [3] and [5]. In particular, it has been shown that it is a theory of massless particles of higher spins. Here, we present a summary of the principal ideas.

The physical Hilbert space in which we are interested is built on the vacuum state of the tensile string theory in the limit $\epsilon \rightarrow 0$. In order to find the mass spectrum in this limit, the momentum p^μ is considered in that limit.

$$\begin{aligned}
p^\mu &= \int_0^{2\pi} d\sigma \mathcal{P}^\mu, & \mathcal{P}^\mu &= \lim_{\epsilon \rightarrow 0} \frac{\dot{X}^\mu}{2\pi\alpha'} = \frac{\dot{X}^\mu}{2\pi c'} \\
&= \frac{1}{2\pi c'} \int_0^{2\pi} d\sigma \left(\sqrt{2c'} B_0^\mu + \sqrt{2c'} \sum_{n \neq 0} B_n^\mu e^{-in\sigma} \right), \\
&= \sqrt{\frac{2}{c'}} B_0^\mu.
\end{aligned} \tag{3.57}$$

Also, it is necessary to note $[B_m^\mu, B_n^\nu] = 0$, thus there is no ordering ambiguity in the operator M_0 and so $a_M = 0$. Therefore, for a state $|\Phi\rangle$ it holds that $M_0 |\Phi\rangle = 0$. Therefore, the mass operator acting on the state is given by [3]

$$\begin{aligned}
m^2 |\Phi\rangle &= -p^\mu p_\mu |\Phi\rangle = -\frac{2}{c'} B_0 \cdot B_0 |\Phi\rangle = \frac{2}{c'} \left(\sum_{m \neq 0} B_{-m} \cdot B_m \right) |\Phi\rangle, \\
&= \frac{2}{c'} \left(\sum_{m \neq 0} \epsilon \eta_{\mu\nu} (\alpha_{-m}^\mu + \bar{\alpha}_m^\mu) (\alpha_{-m}^\nu + \bar{\alpha}_{-m}^\nu) \right) \alpha_{-n_1}^{\mu_1} \dots \alpha_{-n_N}^{\mu_N} \bar{\alpha}_{-m_1}^{\nu_1} \dots \bar{\alpha}_{-m_N}^{\nu_N} |0, k^\mu\rangle_\alpha \\
&= 0.
\end{aligned} \tag{3.58}$$

In the limit ϵ goes to zero, and makes all masses vanish on the induced vacuum $|0, k^\mu\rangle_\alpha = |0, k^\mu\rangle_I$. In addition, we would be able to generate fields of arbitrary spin which are massless. So, it has been shown that the tensionless limit of string theory generates a theory of massless higher spins.

The tensionless vacuum mass can also be obtained using the oscillator modes operators (3.51) [3, 5]:

$$\begin{aligned}
M_0 |0, k^\mu\rangle_I &= 0 = \sum_{m \in \mathbb{Z}} B_{-m} \cdot B_m |0, k^\mu\rangle_I = |0, k^\mu\rangle_I, \\
&= \left(\sum_{m \neq 0} B_{-m} \cdot B_m + B_0^2 \right) |0, k^\mu\rangle_I, \\
&= B_0^2 |0, k^\mu\rangle_I = 0
\end{aligned} \tag{3.59}$$

Therefore, the mass is given by

$$m^2 |0, k^\mu\rangle_I = -p^\mu p_\mu |0, k^\mu\rangle_I = -\frac{2}{c'} B_0^2 |0, k^\mu\rangle_I = 0. \tag{3.60}$$

3.2.5 Induced Vacuum

Now, we will focus our attention on the evolution of the tensile vacuum as we take the tension to zero. First, we will start by finding the tensionless vacuum under the ideas presented in [2]. Then, we link the tensile vacuum in the ultrarelativistic limit with the tensionless vacuum following [2] and [5].

Recalling the physical constraints in (3.51), and noting that the commutativity of B_n^μ in M_0 , then the tensionless vacuum or induced vacuum is given by $M_n |0, k^\mu\rangle_I = 0$, $\forall n \in \mathbb{Z}$. In terms of the oscillator modes, it translates to

$B_n^\mu |0, k^\mu\rangle_I = 0$, $\forall n \in \mathbb{Z}$. This vacuum expression can be written in terms of the new modes $C, \tilde{C}$ defined as [2]

$$C_n^\mu = \frac{1}{2}(A_n^\mu + B_n^\mu), \quad \tilde{C}_n^\mu = \frac{1}{2}(-A_{-n}^\mu + B_{-n}^\mu). \quad (3.61)$$

In this way, $B_n^\mu |0, k^\mu\rangle_I = 0 = (C_n^\mu + \tilde{C}_{-n}^\mu) |0, k^\mu\rangle_I$, for every integer n . These operators form the algebra of two decoupled harmonic oscillators, i.e.

$$[C_m^\mu, C_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} \quad [\tilde{C}_m^\mu, \tilde{C}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}. \quad (3.62)$$

Furthermore, using (3.37) the tensile and tensionless raising and lowering operators are related by

$$\begin{aligned} C_n^\mu &= \frac{1}{2} \left(\sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} \right) \alpha_n^\mu + \frac{1}{2} \left(\sqrt{\epsilon} - \frac{1}{\sqrt{\epsilon}} \right) \bar{\alpha}_{-n}^\mu, \\ \tilde{C}_n^\mu &= \frac{1}{2} \left(\sqrt{\epsilon} - \frac{1}{\sqrt{\epsilon}} \right) \alpha_{-n}^\mu + \frac{1}{2} \left(\sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} \right) \bar{\alpha}_n^\mu \end{aligned} \quad (3.63)$$

As can be seen in (3.63), $C_n, \tilde{C}_n$ contain mixed tensile operators, so that the vacuum in which $C, \tilde{C}$ act is different from the usual tensile theory $|0, k^\mu\rangle_\alpha$. The vacuum $|0, k^\mu\rangle_c$ is defined by

$$C_n^\mu |0, k^\mu\rangle_c = 0 = \tilde{C}_n^\mu |0, k^\mu\rangle_c, \quad \forall n > 0. \quad (3.64)$$

This new vacuum and its excitations can be used as a basis to build the induced vacuum solution to the equation $(C_n^\mu + \tilde{C}_{-n}^\mu) |0, k^\mu\rangle_I = 0$, as was stated in [2].

$$|0, k^\mu\rangle_I = N \prod_{n=1}^{\infty} \exp \left(-\frac{1}{n} C_{-n} \cdot \tilde{C}_{-n} \right) |0, k^\mu\rangle_c. \quad (3.65)$$

We can obtain this vacuum solution also by noting that (3.65) can be thought as a generalization of the squeeze state for a harmonic oscillator. For a given ground state $|0\rangle_1$ of a harmonic oscillator with Hamiltonian H_1 that change instantaneously to H_2 , the ground state can be written as $|0\rangle_1 = e^{-\frac{1}{2} \tanh \beta a_2^\dagger a_2^\dagger} |0\rangle_2$ [25]. Taking into account this fact, we can try to make an ansatz for the induced vacuum considering two decoupled set of oscillators $C_n^{\mu\dagger}, \tilde{C}_n^{\nu\dagger}$:

$$|0, k^\mu\rangle_I = N e^\Gamma |0, k^\mu\rangle_c, \quad \Gamma = \exp \left(\sum_{n=1}^{\infty} f(n) C_n^\dagger \cdot \tilde{C}_n^\dagger \right). \quad (3.66)$$

We substitute the ansatz (3.66) into the equation $B_n^\mu |0, k^\mu\rangle_I = 0$ to obtain

$$\begin{aligned} (C_n + \tilde{C}_n^\dagger) |0, k^\mu\rangle_I &= 0, \\ C_n e^\Gamma |0, k^\mu\rangle_c + \tilde{C}_n^\dagger e^\Gamma |0, k^\mu\rangle_c &= 0, \end{aligned} \quad (3.67)$$

$$[C_n, e^\Gamma] |0, k^\mu\rangle_c + \tilde{C}_n^\dagger e^\Gamma |0, k^\mu\rangle_c = 0, \quad (3.68)$$

After using the identity $[A, e^B] = [A, B]e^B$ if $[[A, B], B] = 0$, we obtain

$$\left[C_n, \sum_{n=1}^{\infty} f(n) C_n^\dagger \cdot \tilde{C}_n^\dagger \right] = \sum_{m=1}^{\infty} f(m) n \delta_{nm} \tilde{C}_m^\dagger = f(n) n \tilde{C}_n^\dagger \quad (3.69)$$

$$\left[f(n)n\tilde{C}_n^\dagger, \sum_{m=1}^{\infty} f(m)C_m^\dagger \cdot \tilde{C}_m^\dagger \right] = 0 \quad (3.70)$$

$$\begin{aligned} & \left[C_n, \sum_{m=1}^{\infty} f(m)C_m^\dagger \cdot \tilde{C}_m^\dagger \right] e^\Gamma |0, k^\mu\rangle_c + \tilde{C}_n^\dagger e^\Gamma |0, k^\mu\rangle_c = 0, \\ & = \left(\sum_{m=1}^{\infty} f(m)[C_n, C_m^\dagger]\tilde{C}_m^\dagger + C_n^\dagger \right) e^\Gamma |0, k^\mu\rangle_c = 0, \\ & = \left(\sum_{m=1}^{\infty} f(m)n\delta_{nm}\eta^{\mu\nu}\eta_{\nu\lambda}\tilde{C}_m^\dagger + C_n^\dagger \right) e^\Gamma |0, k^\mu\rangle_c = 0, \\ & = (f(n)n + 1)\tilde{C}_n^\dagger e^\Gamma |0, k^\mu\rangle_c = 0. \end{aligned} \quad (3.71)$$

Then, $f(n) = -1/n$ and the ansatz corresponds to the vacuum solution (3.65).

Recalling that $B_n^\mu |0, k^\mu\rangle_I = (C_n^\mu + \tilde{C}_{-n}^\mu) |0, k^\mu\rangle_I = 0$, then the induced vacuum is a Neumann boundary state, as was shown in (1.86). Therefore, the vacuum state (3.65) reflects the characteristics of an open string vacuum. This is a remarkable theoretical physics phenomena shown in [2].

On the other hand, we can obtain an explicit relation between the vacuum with Neumann boundary conditions and the tensile string vacuum in the ultrarelativistic limit by noting a Bogoliubov transformation in the the relations (3.63) parameterized by [2]

$$\cosh \theta = \frac{1}{2} \left(\sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} \right), \quad \sinh \theta = \frac{1}{2} \left(\sqrt{\epsilon} - \frac{1}{\sqrt{\epsilon}} \right). \quad (3.72)$$

So, the $\alpha_n = a_n\sqrt{n}$ oscillators can be obtained as an inverse Bogoliubov transformation of the $C_n = b_n\sqrt{n}$ oscillators.

$$\begin{aligned} a_n^\mu &= \cosh \theta b_n^\mu - \sinh \theta \tilde{b}_n^{\mu\dagger}, \\ \tilde{a}_n^\mu &= -\sinh \theta b_n^{\mu\dagger} + \cosh \theta \tilde{b}_n^\mu \end{aligned} \quad (3.73)$$

We can prove that, in addition, the generator A related to such transformation can be used to express the creation-annihilation operators in terms of exponential maps. That is,

$$S = e^A, \quad A = \sum_{n=1}^N \theta b_n^\dagger \cdot \tilde{b}_n^\dagger - \theta b_n \cdot \tilde{b}_n, \quad N \rightarrow \infty \quad (3.74)$$

$$e^A b_n e^{-A} = b_n + [A, b_n] + \frac{1}{2!} [A, [A, b_n]] + \frac{1}{3!} [A, [A, [A, b_n]]] + \dots \quad (3.75)$$

$$\begin{aligned} [A, b_n^\lambda] &= \sum_{m=1}^N \theta \{ [b_m^\dagger \tilde{b}_m^\dagger, b_n] - \theta [b_m \tilde{b}_m, b_n] \} \\ &= \sum_{m=1}^N \theta \tilde{b}_m \cdot [b_m^\dagger, b_n] = -\tilde{b}_n^{\mu\dagger} \eta^{\mu\nu} \eta_{\nu\lambda} \theta \\ &= -\theta \tilde{b}_n^{\lambda\dagger}. \end{aligned} \quad (3.76)$$

In general, for an odd integer or even k commutators, we will obtain the following results in the commutators: k odd $-\theta^k \tilde{b}_n^\dagger$ and k even $\theta^k b_n$. For example,

$$[A, [A, b_n]] = \theta^2 b_n, \quad [A, [A, [A, b_n]]] = -\theta^3 b_n. \quad (3.77)$$

Therefore,

$$\begin{aligned}
e^A b_n e^{-A} &= b_n - \theta \tilde{b}_n^\dagger + \frac{\theta^2}{2!} b_n - \frac{\theta^3}{3!} \tilde{b}_n^\dagger + \frac{\theta^4}{4!} b_n + \dots \\
&= b_n \sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!} - \tilde{b}_n^\dagger \sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!}, \\
&= b_n \cosh \theta - \tilde{b}_n^\dagger \sinh \theta, \\
&= a_n
\end{aligned} \tag{3.78}$$

In the same way, $e^A \tilde{b}_n e^{-A} = \tilde{a}_n = -\sinh \theta b_n^\dagger + \cosh \theta \tilde{b}_n^\dagger$. Therefore, as was shown in [2] and [5], the oscillators modes can be written as

$$\begin{aligned}
a_n^\mu &= S b_n S^{-1} = \cosh \theta b_n^\mu - \sinh \theta \tilde{b}_n^{\mu\dagger}, \\
\tilde{a}_n^\mu &= S \tilde{b}_n S^{-1} = -\sinh \theta b_n^{\mu\dagger} + \cosh \theta \tilde{b}_n^\mu.
\end{aligned} \tag{3.79}$$

Therefore, we can use the operator A to relate the two vacuums through the squeeze operator S [2]:

$$|0, k^\mu\rangle_\alpha = S |0, k^\mu\rangle_c. \tag{3.80}$$

Here, we show a method to simplify (3.80) based on the properties of the algebra $su(1, 1)$ (appendix A). Let's define the operators $K_\pm, K_0$

$$K_+ = \sum_{n=1}^N b_n^\dagger \cdot \tilde{b}_n^\dagger, \quad K_- = \sum_{n=1}^N b_n \cdot \tilde{b}_n, \quad K_0 = \frac{1}{2} \sum_{n=1}^N (b_n^\dagger b_n + \tilde{b}_n^\dagger \tilde{b}_n + \mathbb{I}) \tag{3.81}$$

The operators (3.81) satisfy the algebra of the $su(1, 1)$ group.

$$\begin{aligned}
[K_-, K_+] &= \sum_{n,m} (b_m \tilde{b}_m b_n^\dagger \tilde{b}_n^\dagger - b_n^\dagger \tilde{b}_n^\dagger b_m \tilde{b}_m), \\
&= \sum_{n,m} (b_n^\dagger b_m \tilde{b}_m \tilde{b}_n^\dagger + \delta_{nm} \tilde{b}_m \tilde{b}_n - b_n^\dagger \tilde{b}_n^\dagger b_m \tilde{b}_m), \\
&= \sum_{n,m} (b_n^\dagger b_m \delta_{nm} + \delta_{nm} \tilde{b}_m \tilde{b}_n^\dagger), \\
&= \sum_{n=1}^N (b_n^\dagger b_n + \tilde{b}_n^\dagger \tilde{b}_n + \mathbb{I}) = 2K_0
\end{aligned} \tag{3.82}$$

$$\begin{aligned}
[K_0, K_+] &= \frac{1}{2} \sum_{n,m} (b_n^\dagger b_n + \tilde{b}_n^\dagger \tilde{b}_n + \mathbb{I}) b_m^\dagger \tilde{b}_m^\dagger - b_m^\dagger \tilde{b}_m^\dagger (b_n^\dagger b_n + \tilde{b}_n^\dagger \tilde{b}_n + \mathbb{I}), \\
&= \frac{1}{2} \sum_{n,m} b_n^\dagger b_m^\dagger b_n \tilde{b}_m^\dagger + b_n^\dagger \tilde{b}_m^\dagger \delta_{nm} + b_m^\dagger \tilde{b}_m^\dagger \tilde{b}_n^\dagger \tilde{b}_n - b_m^\dagger \tilde{b}_m^\dagger b_n^\dagger b_n - b_m^\dagger \tilde{b}_m^\dagger \tilde{b}_n^\dagger \tilde{b}_n, \\
&= \sum_{n=1}^N b_n^\dagger \tilde{b}_n^\dagger = K_+
\end{aligned} \tag{3.83}$$

$$[K_0, K_-] = [K_0, K_+]^\dagger = -K_- \tag{3.84}$$

Then, by virtue of the properties of the $su(1,1)$ algebra (Appendix A), S can be written as $S = \exp(\alpha_+ K_+ + \alpha_- K_-)$. Then, the vacuum relation (3.80) translates to

$$\begin{aligned}
|0, k^\mu\rangle_\alpha &= S |0, k^\mu\rangle_c = \exp(\omega K_0 + \alpha_+ K_+ + \alpha_- K_-) |0, k^\mu\rangle_c, \\
&= e^{\gamma_+ K_+} e^{\ln(\gamma_0) K_0} e^{\gamma_- K_-} |0, k^\mu\rangle_c, \\
&= e^{\gamma_+ K_+} e^{\ln(\gamma_0) K_0} |0, k^\mu\rangle_c, \\
&= \exp\left(\tanh\theta \sum_{n=1}^N b_n^\dagger \tilde{b}_n^\dagger\right) \exp\left(\frac{N}{2} \ln(\cosh(\theta)^{-2}) + \frac{1}{2} \sum_{n=1}^N b_n^\dagger b_n + \tilde{b}_n^\dagger \tilde{b}_n\right) |0, k^\mu\rangle_c \\
&= \cosh\theta^{-N} \exp\left(\tanh\theta \sum_{n=1}^N b_n^\dagger \cdot \tilde{b}_n^\dagger\right) |0, k^\mu\rangle_c, \\
&= \left(\frac{1}{\cosh\theta}\right)^N \prod_{n=1}^N \exp\left[\frac{\tanh\theta}{n} C_n^\dagger \cdot \tilde{C}_n^\dagger\right] |0, k^\mu\rangle_c
\end{aligned} \tag{3.85}$$

where we have used the fact that $[b_n^\dagger \cdot \tilde{b}_n^\dagger, b_m^\dagger \cdot \tilde{b}_m^\dagger] = 0$ for $n \neq m$. In the infinite limit of oscillators $N \rightarrow \infty$,

$$\begin{aligned}
|0, k^\mu\rangle_\alpha &= \left(\frac{1}{\cosh\theta}\right)^{1+1+\dots} \prod_{n=1}^{\infty} \exp\left[\frac{\tanh\theta}{n} C_{-n} \cdot \tilde{C}_{-n}\right] |0, k^\mu\rangle_c, \\
&= \sqrt{\cosh\theta} \prod_{n=1}^{\infty} \exp\left(\frac{\tanh\theta}{n} C_{-n} \cdot \tilde{C}_{-n}\right) |0, k^\mu\rangle_c,
\end{aligned} \tag{3.86}$$

where the regularization $1 + 1 + 1 + \dots \infty = \zeta(0) = -1/2$ was used in the last result. For $\epsilon = 1$, the tensile vacuum is recovered in (3.86), i.e. $|0, k^\mu\rangle_\alpha = |0, k^\mu\rangle_C$. In the extreme ultra relativistic limit $\epsilon = 0$,

$$\lim_{\epsilon \rightarrow 0} \sqrt{\cosh\theta} = N, \quad \lim_{\epsilon \rightarrow 0} \tanh\theta = -1. \tag{3.87}$$

$$\Rightarrow |0, k^\mu\rangle_\alpha = \sqrt{\cosh\theta} \prod_{n=1}^{\infty} \exp\left(-\frac{1}{n} C_{-n} \cdot \tilde{C}_{-n}\right) |0, k^\mu\rangle_c \tag{3.88}$$

In this way, we have reached the result (3.88) shown in [2]. Hence, the closed tensile string vacuum $|0, k^\mu\rangle_\alpha$ evolves into the induced vacuum $|0, k^\mu\rangle_I$ in the extreme limit, as can be seen by comparison with (3.65). Since this vacuum is also a Neumann boundary state in this limit, thus the closed string vacuum behaves like an open string free to move in all directions.

Chapter 4

The Tensionless Superstring

4.1 Classical Description

4.1.1 Tensionless Superstring Action

Following the procedure outlined in [17], one can supersymmetrise the tensionless bosonic string. In the tensionless regime, the usual worldsheet metric $h^{\alpha\beta}$ vanishes in the tensionless limit and is replaced by the vector densities $V^\alpha V^\beta$, reflecting the degeneration of the metric. The action for the tensionless limit of the superstring is given by

$$S[V^\alpha, X^\mu, \psi^\mu] = \int d\sigma^2 [V^\alpha V^\beta \partial_\alpha X \cdot \partial_\beta X + \bar{\psi} \cdot \rho^\alpha \partial_\alpha \psi], \quad (4.1)$$

where the fields ψ^μ are two-dimensional Majorana spinors denoted as

$$\psi^\mu = \begin{pmatrix} \psi_0^\mu \\ \psi_1^\mu \end{pmatrix}. \quad (4.2)$$

In order to specify the ρ^α matrices, it is necessary to note the consequences of the metric degeneration of the worldsheet in the Clifford Algebra. As was discussed in the tensionless bosonic theory, the worldsheet metric becomes degenerate and it is replaced by a product of two vector densities

$$T\sqrt{-h}h^{\alpha\beta} \rightarrow V^\alpha V^\beta. \quad (4.3)$$

In light of the change given in (4.3), the Clifford Algebra characterized by $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$ needs to be modified. The metric $\eta^{\alpha\beta}$ is replaced by the product of the vector densities $V^\alpha V^\beta$ (subject to a gauge choice of $V^\alpha = (1, 0)$). Therefore, the tensionless Clifford Algebra is determined by [17]

$$\{\rho^\alpha, \rho^\beta\} = 2V^\alpha V^\beta. \quad (4.4)$$

Then, the relation (4.4) reduces to the following matrix equations:

$$(\rho^0)^2 = \mathbb{1}, \quad (\rho^1)^2 = \mathbb{O}, \quad \rho^0 \rho^1 + \rho^1 \rho^0 = \mathbb{O} \quad (4.5)$$

A set of matrices that satisfies the relations (4.5) is such that

$$\rho^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (4.6)$$

For the previous matrices (4.6), the supersymmetrised action for a fundamental tensionless string can be written as [7]

$$S = \int d\sigma^2 \left[\dot{X}^2 + i(\psi_0 \cdot \dot{\psi}_0 + \psi_1 \cdot \dot{\psi}_1 - \psi_1 \cdot \psi'_0) \right]. \quad (4.7)$$

4.1.2 Symmetries

Just like in the bosonic case, the tensionless superstring action is invariant under local diffeomorphisms and supersymmetry transformations [7].

-*Diffeomorphism invariance*

$$\begin{aligned} \delta X^\mu &= \xi^\alpha \partial_\alpha X^\mu, \\ \delta \psi_0^\mu &= \xi^\alpha \partial_\alpha \psi_0^\mu + \frac{1}{4} \partial_\alpha \xi^\alpha \psi_0^\mu, \\ \delta \psi_1^\mu &= \xi^\alpha \partial_\alpha \psi_1^\mu + \frac{1}{4} \partial_\alpha \xi^\alpha \psi_1^\mu + \frac{1}{2} \partial_1 \xi^0 \psi_0^\mu \end{aligned} \quad (4.8)$$

-*Supersymmetry invariance*

$$\begin{aligned} \delta \psi_0^\mu &= -\epsilon^1 \dot{X}^\mu, \\ \delta \psi_1^\mu &= -\epsilon^0 \dot{X}^\mu - \epsilon^0 X^{\mu'}, \\ \delta X^\mu &= i(\epsilon^0 \psi_0^\mu + \epsilon^1 \psi_1^\mu) \end{aligned} \quad (4.9)$$

By requiring the invariance of $\delta S_\xi = 0$ and $\delta S_\epsilon = 0$, ξ^α , ϵ^α and ψ must satisfy the following conditions [7]:

$$\begin{aligned} \partial_0 \xi^0 &= \partial_1 \xi^1, & \partial_0 \xi^1 &= 0, \\ \partial_0 \epsilon^1 &= \partial_0 \epsilon^0 = \partial_1 \epsilon^1 = \partial_1 \epsilon^0, \\ \partial_0 \epsilon^0 &= \partial_0 \epsilon^1 = 0 \end{aligned} \quad (4.10)$$

$$\psi_0^\mu = \psi_1^{\mu*} \quad (4.11)$$

4.1.3 Equations of Motion and Mode Expansions

The equations of motion obtained by varying (4.1) with respect to X^μ and ψ^μ can be expressed as

$$\ddot{X}^\mu = 0, \quad \dot{\psi}_0^\mu = 0, \quad \dot{\psi}_1^\mu = \psi_0^{\prime\mu}. \quad (4.12)$$

The solutions to the equations (4.12) in the (NS,NS) sector are of the following form

$$\begin{aligned} X^\mu(\tau, \sigma) &= x^\mu + \sqrt{2c'} B_0^\mu \tau + i\sqrt{2c'} \sum_{n \neq 0} \frac{1}{n} (A_n^\mu - in\tau B_n^\mu) e^{-in\sigma}, \\ \psi_0^\mu(\tau, \sigma) &= \sqrt{c'} \sum_{r \in \mathbb{Z}+1/2} \beta_r^\mu e^{-ir\sigma}, & \psi_1^\mu(\tau, \sigma) &= \sqrt{c'} \sum_{r \in \mathbb{Z}+1/2} [\gamma_r^\mu - ir\tau \beta_r^\mu] e^{-ir\sigma}. \end{aligned} \quad (4.13)$$

Here, we have already imposed the closed string boundary conditions on X^μ and ψ^μ in (4.13). As observed, the field X^μ follows the same mode expansion presented in (3.20), while the components of the field ψ^μ is expanded in terms of fermionic

oscillator modes $\beta_r^\mu, \gamma_r^\mu$. From (2.39) and (4.13), we get the non-zero commutations of the fermionic modes

$$\{\gamma_r^\mu, \beta_s^\nu\} = -2i\delta_{r+s}\eta^{\mu\nu}. \quad (4.14)$$

We note that the commutation relations of the oscillators are not in a simple harmonic oscillator basis just as it happened with the tensionless bosonic case in (3.38).

We continue with the classical analysis focusing on the constraint equations for the tensionless superstring. Similarly to the tensile superstring case discussed in the chapter 2, the constraint equations can be worked out by finding the components of the energy-momentum tensor and the supercurrent from the action. These components are given by [7]

$$\dot{X} \cdot X' + \frac{1}{4}(\psi'_0 \cdot \psi_1 + \psi_0 \cdot \psi'_1) = 0, \quad \dot{X}^2 + \frac{i}{2}\psi'_0 \cdot \psi_0 = 0 \quad (4.15)$$

$$\psi_0 \cdot X' + \psi_1 \cdot \dot{X} = 0, \quad \psi_0 \cdot \dot{X} = 0. \quad (4.16)$$

Using the mode expansions (4.13) in the constraint equations (4.15) and (4.16), then they translate to the following expressions

$$\begin{aligned} \dot{X} \cdot X' + \frac{1}{4}(\psi'_0 \cdot \psi_1 + \psi_0 \cdot \psi'_1) &= 4c' \sum_{n \in \mathbb{Z}} (L_n - in\tau M_n) e^{-in\sigma} = 0, \\ \dot{X}^2 + \frac{i}{2}\psi'_0 \cdot \psi_0 &= 4c' \sum_{n \in \mathbb{Z}} M_n e^{-in\sigma} = 0, \\ \psi_0 \cdot X' + \psi_1 \cdot \dot{X} &= 4c' \sum_{r \in \mathbb{Z}+1/2} (G_r - ir\tau H_r) e^{-ir\sigma} = 0, \\ \psi_0 \cdot \dot{X} &= 4c' \sum_{r \in \mathbb{Z}+1/2} H_r e^{-ir\sigma} = 0, \end{aligned} \quad (4.17)$$

where the above terms have been defined as

$$\begin{aligned} L_n &= \frac{1}{2} \sum_m A_{-m} \cdot B_{m+n} + \frac{1}{4} \sum_r (2r+n)(\beta_{-r} \cdot \gamma_{r+n} + \gamma_{-r} \cdot \beta_{r+n}), \\ M_n &= \frac{1}{2} \sum_m B_{-m} \cdot B_{m+n} + \frac{1}{4} \sum_r (2r+n)\beta_{-r} \cdot \beta_{r+n}, \\ G_r &= \frac{1}{2} \sum_m (A_{-m} \cdot \beta_{m+r} + B_{-m} \cdot \gamma_{m+r}), \\ H_r &= \frac{1}{2} \sum_m (B_{-m} \cdot \beta_{m+r}). \end{aligned} \quad (4.18)$$

4.1.4 Super Galilean Conformal Algebra

The classical algebra spanned by the generators (4.18) is known as the classical Super Galilean Conformal Algebra (Inhomogenous) or SGCA_I. By using the brackets relations of the oscillators (3.38) and (4.14), the resulting brackets of the generators

are the following [7]:

$$\begin{aligned}
\{L_m, L_n\} &= -i(m-n)L_{m+n}, & \{L_m, M_n\} &= -i(m-n)M_{m+n}, \\
\{L_m, G_r\} &= -i\left(\frac{m}{2} - r\right)G_{m+r}, & \{M_m, G_r\} &= -i\left(\frac{m}{2} - r\right)H_{m+r}, \\
\{L_m, H_r\} &= -i\left(\frac{m}{2} - r\right)H_{m+r}, & \{G_r, G_s\} &= -2iL_{r+s}, \\
\{G_r, H_s\} &= -2iM_{r+s}.
\end{aligned} \tag{4.19}$$

The SGCA_I is a generalization of the tensionless bosonic algebra or GCA shown in (3.30). This algebra can also be derived by applying an inhomogeneous contraction to the super Virasoro generators shown in (2.36), alongside the contraction in (3.41). The inhomogeneous contraction can be written as [7]

$$G_r = \mathcal{Q}_r - i\bar{\mathcal{Q}}_{-r}, \quad H_r = \epsilon(\mathcal{Q}_r + i\bar{\mathcal{Q}}_{-r}) \tag{4.20}$$

However, from the point of view of the algebra, there are several ways that one could contract the fermionic generators and generalize the GCA. Another consistent algebra is the so called Super Galilean Conformal Algebra (Homogeneous) or SGCA_H . In this case, the SGCA_H can be obtained by a contraction on the super Virasoro generators which scale both in a similar manner [4]. The resulting algebra is less richer than the SGCA_I .

4.1.5 Limit from the Tensile Superstring

Now, we want to find a relation between the tensionless superstring theory and the tensile superstring theory in the UR limit. One way is to reproduce the tensionless action (4.7) starting from the RNS superstring action (4.1) in the UR limit. In order to ensure that our action does not blow up while taking the UR limit, the fermionic degrees of freedom ψ^μ must be scaled.

First, we make a change of variables in the fermionic fields ψ_+^μ and ψ_-^μ given by

$$\psi_0^\mu = \frac{1}{\sqrt{2}}(\psi_+^\mu + i\psi_-^\mu), \quad \psi_1^\mu = \frac{1}{\sqrt{2}}(\psi_+^\mu - i\psi_-^\mu) \tag{4.21}$$

Second, we apply the inhomogeneous scaling to the new fields defined in (4.21). The scaling can be expressed as [7]:

$$\begin{aligned}
\psi_0^\mu &\rightarrow \epsilon\psi_0^\mu, \\
\psi_1^\mu &\rightarrow \psi_1^\mu
\end{aligned} \tag{4.22}$$

The scaling (4.22) together with the UR limit give us the form of the action for the Inhomogeneous tensionless superstring:

$$S = \int d^2\sigma [\dot{X}^2 + i(\psi_0 \cdot \dot{\psi}_1 + \psi_1 \cdot \dot{\psi}_0 - \psi_0 \cdot \psi_0')] \tag{4.23}$$

This is same action (4.7) provided we use the equation (4.11). Therefore, the equations of motion are also the same in the UR limit. Now we can apply the inhomogeneous scaling on the mode expansions of the tensile string [7]. That is,

$$\begin{aligned}
\psi_0^\mu &= \sqrt{c'} \sum_r \sqrt{\epsilon}(b_r^\mu + i\tilde{b}_{-r}^\mu)e^{-ir\sigma}, \\
\psi_1^\mu &= \sqrt{c'} \sum_r \left[\frac{b_r^\mu - i\tilde{b}_{-r}^\mu}{\sqrt{\epsilon}} - ir\tau\sqrt{\epsilon}(b_r^\mu + i\tilde{b}_{-r}^\mu) \right] e^{-ir\sigma}.
\end{aligned} \tag{4.24}$$

The expansion for the scalar field X^μ is given by (3.35). If we compare these modes with the one that we obtained intrinsically in (4.13), it can be seen that

$$\gamma_r^\mu = \frac{1}{\sqrt{\epsilon}}(b_r - i\tilde{b}_{-r}), \quad \beta_r = \sqrt{\epsilon}(b_r + i\tilde{b}_{-r}). \quad (4.25)$$

The remaining relations for the A_n^μ, B_m^ν modes are given by (3.37).

Plugging the relations (4.25) and (3.37) back into the generators (4.18) gives us the connections between the tensile and the tensionless constraints expressed in equations (4.20) and (3.41). Therefore, this inhomogeneous scaling and the UR limit on the tensile superstring is consistent to arrive at SGCA_I .

4.2 Quantum Description

4.2.1 Quantum Constrains

Similar to the bosonic case, the theory of quantum superstrings is formulated in the covariant approach. We promote to operators the constraints (4.18) and then we impose them as physical conditions on the states $|phys\rangle$ of the Hilbert space. These conditions can be written as

$$\begin{aligned} \langle phys|L_n|phys\rangle &= 0, & \langle phys|M_n|phys\rangle &= 0, \\ \langle phys|G_r|phys\rangle &= 0, & \langle phys|H_r|phys\rangle &= 0. \end{aligned} \quad (4.26)$$

In addition, the quantum version of the constraints in (4.18) spans the quantum SGCA_I , and satisfy the following relations [4]

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c_L}{12}(n^3-n)\delta_{n+m}, \\ [L_n, M_m] &= (n-m)M_{n+m} + \frac{c_M}{12}(n^3-n)\delta_{n+m}, \\ [L_n, G_r] &= \left(\frac{n}{2}-r\right)G_{n+r}, & [M_n, G_r] &= \left(\frac{n}{2}-r\right)H_{n+r}, \\ [L_n, H_r] &= \left(\frac{n}{2}-r\right)H_{n+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c_L}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}, \\ \{G_r, H_s\} &= 2M_{r+s} + \frac{c_M}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}. \end{aligned} \quad (4.27)$$

4.2.2 Bogoliubov Transformation on the Worldsheet

As in the tensionless bosonic case, it is convenient to have an oscillator construction of modes. In order to find such oscillator basis, let's define the $C_n^\mu, \tilde{C}_n^\mu$ and $\omega_n^\mu, \tilde{\omega}_n^\mu$ as

$$\begin{aligned} C_n^\mu &= \frac{1}{2}(A_n^\mu + B_n^\mu), & \tilde{C}_n^\mu &= \frac{1}{2}(-A_{-n}^\mu + B_{-n}^\mu), \\ \omega_r^\mu &= \frac{1}{2}(\gamma_r^\mu + \beta_r^\mu), & \tilde{\omega}_r^\mu &= \frac{i}{2}(-\gamma_{-r}^\mu + \beta_{-r}^\mu) \end{aligned} \quad (4.28)$$

The operators presented in (4.28) follow the canonical commutation relations for harmonic oscillators [6]:

$$[C_m^\mu, C_n^\nu] = [\tilde{C}_m^\mu, \tilde{C}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}, \quad \{\omega_r^\mu, \omega_s^\nu\} = \tilde{\omega}_r^\mu, \tilde{\omega}_s^\nu\} = \delta_{r+s}\eta^{\mu\nu} \quad (4.29)$$

In addition, the operators in (4.28) can be splitted out in creation and annihilation operators due to hermiticity. Specifically, the hermiticity of the generators used in (4.27) implies the hermiticity of the modes γ_r and β_r . Furthermore, since we demand $A_m, B_n, \gamma_r, \beta_s$ all to be manifestly hermitian, ω_r is hermitian and $\tilde{\omega}_r$ is anti-hermitian. [6]. Then, the following relations holds

$$\begin{aligned} \gamma_r^\dagger &= \gamma_{-r}, & \beta_r^\dagger &= \beta_{-r}, \\ \omega_r^\dagger &= \omega_{-r}, & \tilde{\omega}_r^\dagger &= -\tilde{\omega}_{-r} \end{aligned} \quad (4.30)$$

Using (4.25), we can write the new operators defined in (4.28) in terms of tensile operators. The resulting expressions are given by

$$\begin{aligned} \omega_r &= b_r \cosh \theta + i\tilde{b}_{-r} \sinh \theta, & C_n^\mu &= \alpha_n^\mu \cosh \theta + \tilde{\alpha}_{-n}^\mu \sinh \theta \\ \tilde{\omega}_r &= -ib_{-r} \sinh \theta + \tilde{b}_r \cosh \theta, & \tilde{C}_n^\mu &= \alpha_{-n}^\mu \sinh \theta + \tilde{\alpha}_n^\mu \cosh \theta, \end{aligned} \quad (4.31)$$

where we again obtain a Bogoliubov transformation between the tensile operators and the new oscillator harmonic basis parameterized by (3.72)

4.2.3 Fermionic Squeezed Vacuum

With the Bogoliubov transformation, we can use the generator A related to such transformation to express the creation-annihilation operators in terms of exponential maps. These relations reads as [6]:

$$\omega_r = S_F^{-1} b_r S_F, \quad \tilde{\omega}_r = S_F^{-1} \tilde{b}_r S_F, \quad (4.32)$$

where

$$S_F = \exp \left(\sum_{r>0} i\theta \omega_{-r} \cdot \tilde{\omega}_{-r} - i\theta \omega_r \cdot \tilde{\omega}_r \right). \quad (4.33)$$

Furthermore, we can now relate the tensile vacuum $|00\rangle_\alpha$, characterized by (2.49), and the vacuum $|00\rangle_C$, which is annihilated under the conditions

$$C_n |00\rangle_C = \tilde{C}_n |00\rangle_C = \omega_r |00\rangle_C = \tilde{\omega}_r |00\rangle_C = 0, \quad \forall n, r > 0 \quad (4.34)$$

The relation is explicitly given by

$$|00\rangle_\alpha = S S_F |00\rangle_C, \quad (4.35)$$

where S is the squeeze operator given by (3.74).

Analogously to the bosonic case, we show a method to simplify (4.35) based on the properties of the algebra $\mathfrak{su}(1, 1)$. Let's define the operators K_+ and K_- as

$$K_+ = i \sum_{r>0} \tilde{\omega}_r^\dagger \cdot \omega_r^\dagger, \quad K_- = i \sum_{r>0} \tilde{\omega}_r \cdot \omega_r \quad (4.36)$$

The remaining generator K_0 can be found using the following commutation relation:

$$\begin{aligned}
[K_-, K_+] &= - \sum_{r,s} (\tilde{\omega}_r^\dagger \omega_r^\dagger \tilde{\omega}_s \omega_s - \tilde{\omega}_s \omega_s \tilde{\omega}_r^\dagger \omega_r^\dagger), \\
&= - \sum_{r,s} (-\tilde{\omega}_r^\dagger \tilde{\omega}_s \omega_r^\dagger \omega_s - \tilde{\omega}_s \omega_s \tilde{\omega}_r^\dagger \omega_r^\dagger), \\
&= - \sum_{r,s} (\tilde{\omega}_s \tilde{\omega}_r^\dagger \omega_r^\dagger \omega_s + \delta_{rs} \omega_r^\dagger \omega_s - \tilde{\omega}_s \omega_s \tilde{\omega}_r^\dagger \omega_r^\dagger), \\
&= - \sum_{r,s} (-\tilde{\omega}_s \tilde{\omega}_r^\dagger \omega_s \omega_r^\dagger - \delta_{rs} \tilde{\omega}_s \tilde{\omega}_r^\dagger + \delta_{rs} \omega_r^\dagger \omega_s - \tilde{\omega}_s \omega_s \tilde{\omega}_r^\dagger \omega_r^\dagger), \\
&= - \sum_{r,s} (\tilde{\omega}_s \omega_s \tilde{\omega}_r^\dagger \omega_r^\dagger - \delta_{rs} \tilde{\omega}_s \tilde{\omega}_r^\dagger + \delta_{rs} \omega_r^\dagger \omega_s - \tilde{\omega}_s \omega_s \tilde{\omega}_r^\dagger \omega_r^\dagger). \\
&= - \sum_{r,s} (\omega_r^\dagger \omega_s - \tilde{\omega}_s \tilde{\omega}_r^\dagger) \delta_{rs} = - \sum_{r>0} (\omega_r^\dagger \omega_r - \tilde{\omega}_r \tilde{\omega}_r^\dagger), \\
&= - \sum_{r>0} (\omega_r^\dagger \omega_s - \tilde{\omega}_s \tilde{\omega}_r^\dagger) \delta_{rs} = - \sum_{r>0} (\omega_r^\dagger \omega_r + \tilde{\omega}_r^\dagger \tilde{\omega}_r + \mathbb{I}), \\
&= 2K_0.
\end{aligned} \tag{4.37}$$

Therefore,

$$K_0 = -\frac{1}{2} \sum_{r>0} (\omega_r^\dagger \omega_r + \tilde{\omega}_r^\dagger \tilde{\omega}_r + \mathbb{I}). \tag{4.38}$$

These operators also satisfy the commutation relations of the $su(1,1)$ algebra.

$$\begin{aligned}
[K_0, K_+] &= -\frac{i}{2} \sum_{r,s} (\tilde{\omega}_r^\dagger \tilde{\omega}_r \tilde{\omega}_s^\dagger \omega_s^\dagger + \omega_r^\dagger \omega_r \tilde{\omega}_s \omega_s^\dagger - \tilde{\omega}_s^\dagger \omega_s^\dagger \tilde{\omega}_r^\dagger \tilde{\omega}_r - \tilde{\omega}_s^\dagger \omega_s^\dagger \omega_r^\dagger \omega_r), \\
&= -\frac{i}{2} \sum_{r,s} (\tilde{\omega}_r^\dagger \omega_s^\dagger \tilde{\omega}_r \tilde{\omega}_s^\dagger + \tilde{\omega}_s^\dagger \omega_r^\dagger \omega_r \omega_s^\dagger - \tilde{\omega}_s^\dagger \omega_s^\dagger \tilde{\omega}_r^\dagger \tilde{\omega}_r - \tilde{\omega}_s^\dagger \omega_s^\dagger \omega_r^\dagger \omega_r), \\
&= -\frac{i}{2} \sum_{r,s} (-\tilde{\omega}_r^\dagger \omega_s^\dagger \tilde{\omega}_s^\dagger \omega_r - \tilde{\omega}_r^\dagger \omega_s^\dagger \delta_{rs} - \tilde{\omega}_s^\dagger \omega_r^\dagger \omega_s^\dagger \omega_r - \tilde{\omega}_s^\dagger \omega_r^\dagger \delta_{sr} \\
&\quad - \tilde{\omega}_s^\dagger \omega_s^\dagger \tilde{\omega}_r^\dagger \tilde{\omega}_r - \tilde{\omega}_s^\dagger \omega_s^\dagger \omega_r^\dagger \omega_r), \\
&= -\frac{i}{2} \sum_{r,s} (\tilde{\omega}_r^\dagger \tilde{\omega}_s^\dagger \omega_s^\dagger \omega_r - \tilde{\omega}_r^\dagger \omega_s^\dagger \delta_{rs} + \tilde{\omega}_s^\dagger \omega_s^\dagger \omega_r^\dagger \omega_r - \tilde{\omega}_s \omega_r^\dagger \delta_{rs} \\
&\quad - \tilde{\omega}_s^\dagger \omega_s^\dagger \tilde{\omega}_r^\dagger \tilde{\omega}_r - \tilde{\omega}_s^\dagger \omega_s^\dagger \omega_r^\dagger \omega_r), \\
&= -\frac{i}{2} \sum_{r>0} -2\tilde{\omega}_r^\dagger \omega_r^\dagger = i \sum_{r>0} \tilde{\omega}_r^\dagger \omega_r^\dagger = +K_+
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
[K_0, K_+]^\dagger &= K_+^\dagger K_0 - K_0^\dagger K_+^\dagger = K_+^\dagger, \\
&= -i \sum_{r>0} \omega_r \tilde{\omega}_r K_0 - K_0 i \sum_{r>0} \omega_r \tilde{\omega}_r, \\
&= i \sum_{r>0} \tilde{\omega}_r \omega_r K_0 + K_0 i \sum_{r>0} \tilde{\omega}_r \omega_r, \\
&= K_- K_0 - K_0 K_- = -[K_0, K_-] = K_-
\end{aligned} \tag{4.40}$$

With the K 's operators defined, we can rewrite the vacuum (4.35) as

$$\begin{aligned}
|00\rangle_\alpha &= S \exp \left(\sum_{r>0} i\theta \omega_{-r} \cdot \tilde{\omega}_{-r} - i\theta \omega_r \cdot \tilde{\omega}_r \right) |00\rangle_C, \\
&= S \exp \left(\sum_{r>0} -i\theta \tilde{\omega}_{-r} \omega_{-r} + i\theta \tilde{\omega}_r \omega_r \right) |00\rangle_C, \\
&= S e^{\alpha_+ K_+ + \alpha_- K_-} |00\rangle_C, \\
&= S e^{\gamma_+ K_+} e^{\ln(\gamma_0) K_0} e^{\gamma_- K_-} |00\rangle_C,
\end{aligned} \tag{4.41}$$

where $\omega = 0, \alpha_+ = -\theta, \alpha_- = 0$, and

$$\begin{aligned}
z^2 &= \frac{\omega^2}{4} - \alpha_+ \alpha_- = \theta^2, \\
\gamma_\pm &= \frac{2\gamma_\pm \sinh(z)}{2z \cosh(z) - \omega \sinh(z)} = \mp \tanh(\theta), \\
\gamma_0 &= (\cosh(z) - \frac{\omega}{2z} \sinh(z))^{-2} = \cosh(\theta)^{-2}
\end{aligned} \tag{4.42}$$

In this way, we finally obtain

$$\begin{aligned}
|00\rangle_\alpha &= S e^{-\tanh(\theta) K_+} e^{-2 \ln(\cosh(\theta)) K_0} e^{\tanh(\theta) K_-} |00\rangle_C, \\
&= S e^{-\tanh(\theta) K_+} e^{-2 \ln(\cosh \theta) K_0} |00\rangle_C, \\
&= S e^{-\tanh(\theta) K_+} \exp \left(\ln(\cosh \theta) \sum_{r>0} (\omega_r^\dagger \omega_r + \tilde{\omega}_r^\dagger \tilde{\omega}_r + \mathbb{I}) \right) |00\rangle_C, \\
&= S e^{-\tanh(\theta) K_+} e^{N \ln(\cosh \theta)} \exp \left(\sum_{r>0} (\omega_r^\dagger \omega_r + \tilde{\omega}_r^\dagger \tilde{\omega}_r) \right) |00\rangle_C, \\
&= S \cosh(\theta)^N \exp \left(-i \tanh(\theta) \sum_{r>0} \tilde{\omega}_r^\dagger \omega_r^\dagger \right) |00\rangle_C, \\
&= \exp \left(\sum_{n=1} \frac{\tanh \theta}{n} C_n^\dagger \cdot \tilde{C}_n^\dagger \right) \exp \left(-i \tanh(\theta) \sum_{r>0} \tilde{\omega}_r^\dagger \omega_r^\dagger \right) |00\rangle_C, \\
&= \prod_{n,r>0} \exp \left[\tanh \theta \left(\frac{C_n^\dagger \cdot \tilde{C}_n^\dagger}{n} + i \omega_r^\dagger \cdot \tilde{\omega}_r^\dagger \right) \right] |00\rangle_C
\end{aligned} \tag{4.43}$$

In the extreme limit $\epsilon = 0$, $\tanh \theta = -1$. Then,

$$|00\rangle_\alpha = \prod_{n,r>0} \exp \left[- \left(\frac{C_n^\dagger \cdot \tilde{C}_n^\dagger}{n} + i \omega_r^\dagger \cdot \tilde{\omega}_r^\dagger \right) \right] |00\rangle_C \tag{4.44}$$

This result is in agreement with the vacuum shown in [6]. As in the tensile fermionic vacuum shown in (2.52), the vacuum (4.44) incorporates contributions from both fermionic and bosonic creation operators, which behave as harmonic oscillators. Nevertheless, the functional form differs significantly. Notably, the vacuum exhibits an exponential decrease with respect to the positive integers n in the bosonic contribution. Interestingly, unlike the tensionless bosonic vacuum described in (3.88), the tensionless limit here does not introduce an infinite normalization factor outside the exponential as we take the tensionless limit.

Chapter 5

Conclusions and Future Directions

In this thesis project, we have explored the theory of tensionless strings. The investigation focused on the symmetries of the string worldsheet, treating the tensionless strings both as a limit of the conventional tensile theory and as a fundamental object in its own right.

It has been shown that the equations of motion, mode expansions, and constraint equations for the tensionless bosonic string and the inhomogeneous tensionless superstring are consistent with the Galilean Conformal Algebra (GCA) and the Super Galilean Conformal Algebra (SGCA), respectively [1, 3, 4, 7]. Additionally, significant differences emerged when studying the quantum tensionless closed string using canonical quantization. For the bosonic case, imposing quantum constraints lead to distinct quantum mechanical theories[5]. By examining the induced vacuum through worldsheet Bogoliubov transformations, we identified this vacuum as a Neumann boundary state, indicating the emergence of open strings from closed strings in the tensionless limit [3, 2]. Furthermore, the masses of all these states were found to vanish [3].

For the tensionless superstring, we analyzed the inhomogeneous tensionless superstring, characterized by the residual symmetry algebra $SGCA_I$ [7]. In the tensionless limit, the degeneration of the worldsheet metric modifies the Clifford algebra for worldsheet fermions to reflect this feature. Bogoliubov transformations connecting tensile and tensionless oscillators were constructed, establishing a mapping between the two superstring vacua[7, 6].

It was found that the tensionless superstring vacuum in the (NS,NS) sector, unlike the bosonic vacuum, incorporates fermionic harmonic oscillator operators in addition to the bosonic ones. Furthermore, this tensionless superstring vacuum exhibits a notably different functional form compared to the superstring vacuum in conventional tension theory. This discrepancy with the usual tension string theory is also observed in the tensionless bosonic vacuum, which presents an exponential functional form with bosonic creation operators. Another striking feature of the tensionless superstring vacuum is that there is no normalization constant that diverges outside the exponential operator, as occurs in the case of the bosonic vacuum when the tensionless limit is taken. [6, 2]

Furthermore, when deriving the tensionless bosonic vacuum [2] and the inhomogeneous tensionless superstring vacuum [6], a method based on the properties of the $su(1,1)$ algebra was proposed, providing a structured approach for their computation.

In future work, the relationship between infinite distance limits and the emergence of tensionless strings will be examined. Infinite distance limits are characterized by qualitative transitions in physical theories. A key aspect in the exploration of such limits will involve the information metric, a powerful tool that establishes a notion of distance within any continuous family of theories. In the context of string theory, infinite distance limits correspond to the emergence of a tower of light states that become progressively lighter, as illustrated in the mass spectrum of the tensionless bosonic string, for instance, in equation (3.58). In infinite distance limits, tensionless strings represent a concrete physical realization of the tower of light states predicted by the swampland conjecture [22].

The classical information metric quantifies the separation between probability distributions by measuring how easily they can be distinguished. This concept is extended by the quantum information metric, which defines a measure of distance in the space of quantum states, often evaluated through the vacuum states. Within quantum theories, this metric captures the differences between neighboring theories by assessing the distinguishability of their predictions, placing them further apart when they are more easily differentiable. [21].

Additionally, infinite distance limits are linked to factorization of expectation values. As discussed in [22], this refers to a phenomenon where the information metric exhibits an infinite distance singularity. This occurs when the expectation values of observables factorize, implying the disappearance of statistical or quantum correlations between degrees of freedom. In such scenarios, the system becomes effectively simpler or more deterministic, and the ambiguity inherent to sampling from the probability distribution vanishes.

The connection between infinite distance factorization and tensionless string theory, as noted in [22], highlights the universality of this feature in specific physical configurations, including string theory.

As an example, consider the distance in the quantum harmonic oscillator with frequency ω :

$$d(\omega_1, \omega_0) = \log \left(\frac{\omega_1}{\omega_0} \right) \quad (5.1)$$

In the limit $\omega \rightarrow \infty$, the particle is confined to the minimum of the potential, making the ground state probability distribution to collapse to a single point. In contrast, as $\omega \rightarrow 0$, the particle behaves as free, and the eigenfunctions lose their smoothness and normalizability. In both of these extreme limits, the distance (5.1) diverges. These infinite distance limits in the information metric correspond to theories where expectation values factorize: $\langle x^n \rangle = \langle x \rangle^n$. However, the explicit physical observables from tensionless strings that undergo factorization in this context remain unidentified. Exploring such observables, particularly for the tensionless bosonic string and tensionless superstring in the (NS, NS) sector, would provide valuable insights into the underlying mechanics of tensionless strings and their relation to infinite distance limits.

Appendix A

$su(1,1)$ Algebra

The elements g of the $SU(1,1)$ group satisfies

$$gSg^\dagger = S. \quad (\text{A.1})$$

The generators of the group are operators K_i , $i = 0, 1, 2$, that satisfy the commutation relations

$$[K_1, K_2] = -iK_0, \quad [K_2, K_0] = iK_1, \quad [K_0, K_1] = iK_2. \quad (\text{A.2})$$

The resulting algebra $su(1,1)$ introduce the raising and lowering operators $K_\pm$ in the standard form

$$K_\pm = K_1 \pm iK_2, \quad K_+ = K_-^\dagger, \quad (\text{A.3})$$

which obey the following relations: $[K_0, K_\pm] = \pm K_\pm$, $[K_-, K_+] = 2K_0$.

The properties of the group and its algebra allow to obtain a decomposition formulae of the generic element [\[16\]](#)

$$\exp(\omega K_0 + \alpha_+ K_+ + \alpha_- K_-) = \exp(\gamma_+ K_+) \exp(\ln(\gamma_0) K_0) \exp(\gamma_- K_-) \quad (\text{A.4})$$

where

$$\gamma_0 = \left(\cosh \Xi - \frac{\omega}{2\Xi} \sinh \Xi \right)^{-2}, \quad \gamma_\pm = \frac{2\alpha_\pm \sinh \Xi}{2\Xi \cosh \Xi - \omega \sinh \Xi}, \quad \Xi^2 = \frac{\omega^2}{4} - \alpha_+ \alpha_-.$$

Appendix B

Conformally flat 2D metric

Any two-dimensional Riemannian manifold (M, g) is conformally flat. Let (x, y) be the original local coordinates with which the metric takes the form

$$ds^2 = g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2 \quad (\text{B.1})$$

$$ds^2 = \left(\sqrt{g_{xx}}dx + \frac{g_{xy} + i\sqrt{g}}{\sqrt{g_{xx}}}dy \right) \left(\sqrt{g_{yy}}dx + \frac{g_{xy} - i\sqrt{g}}{\sqrt{g_{xx}}}dy \right), \quad (\text{B.2})$$

where $g = g_{xx}g_{yy} - g_{xy}^2$. According to the theory of differential equations, there exists an integrating factor $\lambda(x, y) = \lambda_1(x, y) + i\lambda_2(x, y)$ such that

$$\begin{aligned} \lambda \left(\sqrt{g_{xx}}dx + \frac{g_{xy} + i\sqrt{g}}{\sqrt{g_{xx}}}dy \right) &= du + idv, \\ \bar{\lambda} \left(\sqrt{g_{yy}}dx + \frac{g_{xy} - i\sqrt{g}}{\sqrt{g_{xx}}}dy \right) &= du - idv, \end{aligned} \quad (\text{B.3})$$

Then $ds^2 = (du^2 + dv^2)/|\lambda|^2$ and by setting $|\lambda|^{-2} = e^{2\Lambda}$, we have the desired coordinate system. If (M, g) is a Lorentz manifold, we have integrating factors $\lambda(x, y)$ and $\mu(x, y)$ such that

$$\begin{aligned} \lambda \left(\sqrt{g_{xx}}dx + \frac{g_{xy} + \sqrt{-g}}{\sqrt{g_{xx}}}dy \right) &= du + dv, \\ \mu \left(\sqrt{g_{yy}}dx + \frac{g_{xy} - \sqrt{-g}}{\sqrt{g_{xx}}}dy \right) &= du - dv, \end{aligned} \quad (\text{B.4})$$

In terms of the coordinates (u, v) the metric takes the form $ds^2 = \lambda^{-1}\mu^{-1}(du^2 - dv^2)$. The product $\lambda\mu$ is either positive definite or negative definite and we may set $1/|\lambda\mu| = e^{2\Lambda}$ to obtain the form [18]

$$ds^2 = \pm e^{2\Lambda}(du^2 - dv^2) \quad (\text{B.5})$$

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