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Various Proofs of The Fundamental Theorem Of Algebra

Proyecto de Investigación

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Various Proofs of The Fundamental Theorem of Algebra

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RESUMEN

El Teorema Fundamental del Álgebra establece que un polinomio no-constante con coeficientes complejos tiene al menos una raíz dentro de los números complejos. En el presente trabajo de invistigación se presentan pruebas del teorema utilizando distintas ramas de la matemática: cálculo multivariable, algebra lineal y análisis complejo.

Palabras Clave: Teorema Fundamental del Álgebra, Teorema del Valor Extremo, Teorema de Fubini, Algebra Lineal, Teorema del Valor Intermedio, Teorema de Liouville.

ABSTRACT

The Fundamental Theorem of Algebra states that a non-constant polynomial with complex coefficients has at least one complex root. Throughout this research paper we will present several proofs using distinct branches of mathematics, such as multivariable calculus, linear algebra and complex analysis.

Keywords: Fundamental Theorem of Algebra, Extreme Value Theorem, Fubini's Theorem, Linear Algebra, Middle Value Property, Liouville's Theorem.

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INTRODUCTION

The fundamental theorem of algebra states that given any polynomial of degree greater or equal to 1, it has at least one complex root. This is a simple statement to understand, but its consequences are not trivial. The use of polynomials and the complex field span most of the mathematical disciplines that the students of Universidad San Francisco are presented with during their undergraduate studies. This is why it should come as no surprise that when one attempts to prove this theorem, it is possible to do so through multiple branches of mathematics. Historically, a first proof was outlined, but not completed by D'Alambert; but the German mathematician Carl Frederich Gauss was the first person who completed a proof. Since then, multiple proofs have been presented. This is a research paper in which the goal is to explore this various proofs. tthis in he following pages we shall show proofs using different branches of mathematics, such as multivariable calculus, linear algebra, and complex analysis.

Proving the F.T.A.

2.1 Calculus EVT proof

This proof is based on the original proof by Lankham, Nachtergaele and Schilling from the University of California [1].

We shall use the Extreme Value Theorem, stated as follows:

Remark. *Let $f : D \longrightarrow \mathbb{R}$ be a continuous function on the closed disk $D \subset \mathbb{R}$. Then f is bounded and attains its minimum and maximum in D.*

We shall also use the following facts:

Remark. *Polynomials are continuous functions.*

Remark. *Modulus is a continuous function*

Remark. *The composition of continuous functions is continuous.*

Theorem 2.1.1. *The Fundamental Theorem of Algebra*
Given any positive integer $n \geq 1$ and any choice of complex numbers $a_0, a_1, ..., a_n$, such that $a_n \neq 0$, the polynomial equation

$$a_n z^n + ... + a_1 z + a_0 = 0 \tag{2.1}$$

has at least one solution in $\mathbb{C}$

Proof. Since polynomials and the modulus are continuous functions, and the composition of continuous functions is continuous, then given a polynomial

$$f(z) = a_n z^n + ... + a_1 z + a_0 \quad , a_i \in \mathbb{C}, \quad a_n \neq 0 \tag{2.2}$$

we may define a new real-valued function

$$|f(z)| \quad \forall z \in \mathbb{C} \tag{2.3}$$

which will be continuous.

Since function 2.3 is continuous, it will be bounded and attain its minimum and maximum on the closed disk D where it's defined by remark 2.1.

Let z_0 be the point where said minimum is attained and suppose $f(z_0) \neq 0$. Then we may define another function:

$$g(z) = \frac{f(z+z_0)}{f(z_0)} \quad \forall z \in \mathbb{C} \tag{2.4}$$

Writing $g(z)$ out we get

$$g(z) = b_n z^n + \ldots + b_k z^k \ldots + b_1 z + 1 \quad \forall z \in \mathbb{C} \tag{2.5}$$

which is a polynomial of degree n with minimum modulus at $z = 0$ and $g(0) = 1$

Taking $n \geq 1$ and $b_k \neq 0$ for some $1 \leq k \leq n$, let $b_k = |b_k| e^{i\theta}$ and $z = r|b_k|^{\frac{-1}{k}} e^{\frac{i(\pi-\theta)}{k}}$.

For $r \geq 0$

$$g(z) = 1 - r^k + r^{k+1} h(r) \tag{2.6}$$

where $h(r)$ is a polynomial.

For $r \leq 1$ we may apply the triangle inequality:

$$|g(z)| \leq 1 - r^k + r^{k+1} |h(r)| \tag{2.7}$$

For a small enough value of r we have $r|h(r)| < 1$ since $rh(r)$ is continuous and $r(h(r)) = 0$ for $r = 0$.

Thus

$$|g(z)| \leq 1 - r^k (1 - r|h(r)| < 1 \tag{2.8}$$

for some r of the form $(0, r_0)$ and $r_0 > 0$.

Thus 1 is not the minimum value of $g(z)$.

The minimum $|f(z_0)| \neq 0$ if and only if the minimum modulus of $g(z) = 1$. Since the latter doesn't hold, the former fails as well; thus proving theorem 2.1.1

■

2.2 Calculus: Fubini's Theorem

This proof was presented by Keith Conrad, in his expository papers from the University of Connecticut [2].

We shall use Fubini's Theorem, stated as follows:

Remark. *Let* $[a,b]\times[c,d]\in\mathbb{R}^2$ *be a rectangle in* $\mathbb{R}^2$*, X=[a,b], Y=[c,d]. Then given a continuous function* f *with real values defined in this rectangle we have:*

$$\int_X\left(\int_Y f(x,y)dy\right)dx=\int_Y\left(\int_X f(x,y)dx\right)dy \tag{2.9}$$

Theorem 2.2.1. *The Fundamental Theorem of Algebra*
Any polynomial of the form

$$f(z)=z^n+\ldots+a_1z+a_0 \quad n\geq 1 \quad a_i\in\mathbb{C} \tag{2.10}$$

has a complex root.

Proof. Suppose

$$f(z)=z^n+\ldots+a_1z+a_0 \tag{2.11}$$

has no complex roots.

We then write the substitution

$$z=re^{i\theta} \quad \text{thus} \quad z^j=r^jcos(j\theta)+ir^jsin(j\theta) \tag{2.12}$$

With this, f is restated as follows:

$$f(z)=r^ncos(n\theta)+\ldots+\mathrm{Re}(a_0)+ir^nsin(n\theta)+\ldots+i\mathrm{Im}(a_0) \tag{2.13}$$

Separating f into it's real and imaginary parts we obtain:

$$f(z)=P(r,\theta)+iQ(r,\theta) \tag{2.14}$$

P and Q are polynomials over $\mathbb{R}$ of degree n with constant terms a_jr^j independent of θ. By this fact we have

$$P(0,\theta)=\mathrm{Re}(a_0) \quad \text{and} \quad Q(0,\theta)=\mathrm{Im}(a_0) \quad ,\forall\theta \tag{2.15}$$

From this observation:

$$\left.\frac{\partial P}{\partial\theta}\right|_{r=0}=0 \quad \text{and} \quad \left.\frac{\partial Q}{\partial\theta}\right|_{r=0}=0 \tag{2.16}$$

Also P and Q are 2π-periodic, as are $\dfrac{\partial P}{\partial r}$ and $\dfrac{\partial Q}{\partial r}$.

To say that the function 2.11 has no complex roots is the same as saying P and Q are not simultaneously *zero* anywhere.

We may write 2.14 in polar coordinates and analyze it's angular component or argument U.

$$U = \arctan\left(\frac{Q}{P}\right) \tag{2.17}$$

The derivatives of 2.17 with respect to r and θ are 2.18 and 2.19 respectively.

$$\frac{\partial U}{\partial r} = \frac{1}{\left(\frac{P}{Q}\right)^2 + 1} \cdot \frac{P\frac{\partial Q}{\partial r} - Q\frac{\partial P}{\partial r}}{P^2} \tag{2.18}$$

$$\frac{\partial U}{\partial r} = \frac{P\frac{\partial Q}{\partial r} - Q\frac{\partial P}{\partial r}}{P^2 + Q^2}$$

$$\frac{\partial U}{\partial \theta} = \frac{P\frac{\partial Q}{\partial \theta} - Q\frac{\partial P}{\partial \theta}}{P^2 + Q^2} \tag{2.19}$$

where $P^2 + Q^2 \neq 0$ by our hypothesis.
It is also clear that

$$\frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial \theta}\right) \tag{2.20}$$

thus we may apply the equation 2.9 as follows:

$$\int_0^R \left(\int_0^{2\pi} \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial \theta}\right) d\theta\right) dr = \int_0^{2\pi} \left(\int_0^R \frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial r}\right) dr\right) d\theta \tag{2.21}$$

For the left-hand side

$$\int_0^R \left(\int_0^{2\pi} \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial \theta}\right) d\theta\right) dr \tag{2.22}$$

$$= \int_0^R \left(\int_0^{2\pi} \frac{\partial}{\partial r} \left(\frac{\partial U}{\partial \theta} \right) d\theta \right) dr$$

$$= \int_0^R \left(\int_0^{2\pi} \frac{\partial}{\partial \theta} \left(\frac{\partial U}{\partial r} \right) d\theta \right) dr$$

$$= \int_0^R \left(\left. \frac{\partial U}{\partial r} \right|_{\theta=0}^{\theta=2\pi} \right) dr$$

$$= \int_0^R (0)\, dr = 0$$

since the function $\frac{\partial U}{\partial r}$ is 2π periodic.

For the right-hand side

(2.23)
$$\int_0^{2\pi} \left(\int_0^R \frac{\partial}{\partial r} \left(\frac{\partial U}{\partial \theta} \right) dr \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_0^R \frac{\partial}{\partial r} \left(\frac{\partial U}{\partial \theta} \right) dr \right) d\theta$$

$$= \int_0^{2\pi} \left(\left. \frac{\partial U}{\partial \theta} \right|_{r=0}^{r=R} \right) d\theta$$

Thus we are left with

(2.24)
$$\int_0^{2\pi} \left(\frac{\partial U}{\partial \theta}(R,\theta) - \frac{\partial U}{\partial \theta}(0,\theta) \right) d\theta$$

for the right hand side.
Writing out the partial derivatives of P and Q

(2.25)
$$\frac{\partial P}{\partial \theta} = -nr^n sin(n\theta) + \ldots$$

(2.26)
$$\frac{\partial Q}{\partial \theta} = nr^n cos(n\theta) + \ldots$$

The numerator of $\frac{\partial U}{\partial \theta}$ is

$$P\left(\frac{\partial Q}{\partial \theta}\right)-Q\left(\frac{\partial P}{\partial \theta}\right) \tag{2.27}$$

$$= nr^{2n}cos^2(n\theta)+...+nr^{2n}sin^2(n\theta)+...+r^2cos^2(\theta)+r^2sin^2(\theta)$$

$$= nr^{2n}\left(cos^2(n\theta)+sin^2(n\theta)\right)+..+r^2\left(cos^2(\theta)+sin^2(\theta)\right).$$

$$= nr^{2n}+(n-1)r^{2(n-1)}...+r^2$$

On the other hand, the denominator of $\frac{\partial U}{\partial \theta}$ is

$$P^2+Q^2 = r^{2n}cos^2(n\theta)+...+(\mathrm{Re}(a_0))^2+r^{2n}sin^2(n\theta)+...+(\mathrm{Im}(a_0))^2 \tag{2.28}$$

$$= r^{2n}\left(cos^2(n\theta)+sin^2(n\theta)\right)+...+(\mathrm{Re}(a_0))^2+(\mathrm{Im}(a_0))^2$$

$$= r^{2n}+...+(\mathrm{Re}(a_0))^2+(\mathrm{Im}(a_0))^2$$

Using 2.27 and 2.28

$$\frac{\partial U}{\partial \theta} = \frac{nr^{2n}+...+r^2}{r^{2n}+...+(\mathrm{Re}(a_0))^2+(\mathrm{Im}(a_0))^2} \tag{2.29}$$

Since we wish to prove theorem 2.2.1 for all $\mathbb{C}$, we take the limit when $R\to\infty$ from 2.29.

$$\lim_{R\to\infty}\frac{\partial U}{\partial \theta}(r=R,\theta) \tag{2.30}$$

$$= \lim_{R\to\infty}\frac{nR^{2n}+...+R^2}{R^{2n}+...+(\mathrm{Re}(a_0))^2+(\mathrm{Im}(a_0))^2}$$

$$= n$$

Thus putting this result into 2.24

$$\lim_{R\to\infty}\int_0^{2\pi}\left(\frac{\partial U}{\partial \theta}(R,\theta)-\frac{\partial U}{\partial \theta}(0,\theta)\right)d\theta \tag{2.31}$$

$$= \int_0^{2\pi}(n)\,d\theta$$

$$= 2\pi n$$

For the left hand-side 2.22 and the right hand-side 2.24 of the equation to be equal

$$2\pi n = 0 \tag{2.32}$$

This only holds if $n = 0$, which is a contradiction, thus proving theorem 2.2.1.

■

2.3 Linear Algebra proof

The proof using linear algebra was also presented by Conrad and can be found in the digital repository of the University of Connecticut [3].

Theorem 2.3.1. *The Fundamental Theorem of Algebra*
Any polynomial of the form

$$f(z) = a_n z^n + ... + a_1 z + a_0 \quad n \geq 1 \quad a_n \neq 0 \quad a_0 \neq 0 \tag{2.33}$$

has a complex root.

To reformulate theorem 2.3.1 in terms of linear algebra we need to rewrite $f(z)$.
One can prove by induction that the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

satisfies

$$\det(\lambda I_n - A) = f(\lambda) \tag{2.34}$$

Equation 2.34 is also known as the characteristic polynomial of A. The zeros of the characteristic polynomial are the eigenvalues of the matrix.

Thus theorem 2.3.1 can be restated as follows:

Theorem 2.3.2. *For each $n \geq 1$, every $n \times n$ square matrix over $\mathbb{C}$ has an eigenvector.*

Note that $a_0 \neq 0$, because that would make zero a root. This is equivalent to saying A would be singular.

Usually, in any Linear Algebra class, the existence of eigenvalues is proved by using theorem 2.3.1; by the fact that every characteristic polynomial has a complex root. But in this case we shall prove it the other way around.

First we prove the following Lemma:

Lemma 2.3.3. *Fix an integer $m > 1$. Suppose that $\forall\ \mathbb{F}-vector$ spaces V whose dimension is not divisible by m, every linear operator on V has an eigenvector. Then for every V where $m \nmid dim(V)$, any pair of commuting linear operators on V has a common eigenvector.*

Proof. This is done by the use of strong induction on the dimension d of the vector space V where d is not divisible by m.

i Basis step: $d = 1$

When $d = 1$, V is a one-dimensional vector space in which any nonzero vector is an eigenvector of any linear operator and any two linear operators commute.

ii Inductive step:

Assume that $d > 1$ is not divisible by m. Let A_1 and A_2 be commuting linear operators on V. Suppose A_1 has an eigenvalue $\lambda \in \mathbb{C}$.

We have the subspaces

$$U = im(A_1 - \lambda I_V) \tag{2.35}$$

and

$$W = ker(A_1 - \lambda I_V) \tag{2.36}$$

of V.

U and W are closed over A_1, i.e.

$$u \in U \Rightarrow A_1(u) \in U \quad \text{and} \quad w \in W \Rightarrow A_1(w) \in W \tag{2.37}$$

Since if $u \in U$

$$u = (A_1 - \lambda I_V)v$$

$$A_1 u = A_1(A_1 - \lambda I_V)v$$

$$A_1 u = A_1(A_1 v - \lambda I_V v)$$

$$A_1 u = A_1(A_1 v) - \lambda A_1 v)$$

$$A_1 u = (A_1 - \lambda)A_1 v$$

Since $A_1 v \in \mathbb{C}$,

$$A_1 u = (A_1 - \lambda)v^* \quad \in U$$

And since if $w \in W$

$$(A_1 - \lambda I_W)w = 0$$

$$A_1 w - \lambda I_W w = 0$$

$$A_1 w - \lambda w = 0$$

$$A_1 w = \lambda w$$

$$(A_1 - \lambda I_W)A_1 w = \lambda(A_1 - \lambda I_W)w = 0$$

Thus

$$A_1 w \in W$$

Also note that

(2.38) $$dim(W) > 0$$

since λ is an eigenvalue of A_1.

We know that $dim(U) + dim(W) = dim(V) = d$ which is not divisible by m. Which leaves us with two cases:

Case 1: Given $dim(U) > 0$, one of the subspaces U or W must have a dimension not divisible by m and be a proper subspace of V. This implies it's dimension is smaller than d and, by the strong induction hypothesis, has the property that in that subspace A_1 and A_2 have a common eigenvector, thus in V, they do too.

Case 2: $dim(U) = 0$ and $dim(W) = d$. In this case every vector in V is an eigenvector for A_1, thus the eigenvector of A_2 is also an eigenvector of A_1.

■

Corollary 2.3.3.1. *For every real vector space V whose dimension is odd, any pair of commuting linear operators have a common eigenvector.*

To prove the corollary we need to know the existence of eigenvectors in spaces with odd dimensions.

For this we will use the Intermediate Value Theorem stated as follows:

Remark. *Given a continuous function f and an interval $[a,b]$, assume without loss of generality that $f(a) \leq f(b)$, then there exists some point c such that $a \leq c \leq b$ such that $f(a) \leq f(c) \leq f(b)$ for all such f(c).*

Lemma 2.3.4. *Every polynomial with real coefficients of odd degree has at least one root.*

Proof. Let

$$f(z) = a_n z^n + \ldots + a_1 z + a_0 \quad n \geq 1 \quad a_n \neq 0 \quad a_i \in \mathbb{R} \tag{2.39}$$

be a polynomial of odd degree.

For the case where $a_n > 0$ we have:

Taking the limits at the extremes

$$\lim_{z \to \infty} f(z) = +\infty \tag{2.40}$$

and

$$\lim_{z \to -\infty} f(z) = -\infty \tag{2.41}$$

Thus there exists some R and P in $\mathbb{R}$ such that

$$\forall r > R \quad f(r) > 0 \tag{2.42}$$

and

$$\forall p < P \quad f(p) < 0 \tag{2.43}$$

Applying theorem 2.3, there must exist a c such that

$$P < c < R \tag{2.44}$$

and

$$f(c) = 0 \tag{2.45}$$

For the case where $a_0 < 0$ the proof is done in a similar manner, with a sign change on the results of the limits. ■

By lemma 2.3.4, we know matrices of odd degree have at least one eigenvector. Then, taking lemma 2.3.3 with $\mathbb{F} = \mathbb{R}$ and $m = 2$ we prove corollary 2.3.3.1.

Next we shall use all of the above to prove the Fundamental Theorem of Algebra.

Proof. of theorem 2.3.1. We will also use strong induction, as in the last proof.

Take 2^k to be the highest power of 2 that divides n, i.e, $n = 2^k n'$, where $2 \nmid n'$.

i Basis step: $k = 0$, which means n is odd.

This was proved in lemma 2.3.4 for matrices with real-valued entries.

For the complex-valued case we pick $A \in M_n(\mathbb{C})$

The set of all Hermitan matrices is defined as

$$H_n = \{\in M_n(\mathbb{C}) | M^* = M\} \tag{2.46}$$

where $M^* = \bar{M}^T$, the conjugate-transpose.

They form a real vector space with dimension n^2 over $\mathbb{R}$. (They do not form a vector space over $\mathbb{C}$ since iI_n is not Hermitian). Since n is odd, they form a vector space of odd dimension over $\mathbb{R}$.

Take a matrix $C \in M_n(\mathbb{C}) = AB$, $A, B \in M_n(\mathbb{C})$ where A is any matrix and B is Hermitian.

Note: we are proving A has an eigenvector, not C.

Since any arbitrary matrix C can be written as the sum of a Hermitian matrix and a skew-Hermitian matrix as follows

$$C = \frac{C + C^*}{2} + \frac{C - C^*}{2} \tag{2.47}$$

Then

$$AB = \frac{AB + B^* A^*}{2} + i \frac{AB - B^* A^*}{2i} \tag{2.48}$$

Since $B \in H_n$ 2.48 becomes:

(2.49) $$AB = \frac{AB + BA^*}{2} + i\frac{AB - BA^*}{2i}$$

We then define the $\mathbb{R}$-linear operators $L_1, L_2 : H_n \to H_n$ as

(2.50) $$L_1(B) = \frac{AB + B^*A^*}{2} \quad L_2(B) = \frac{AB - B^*A^*}{2i}$$

The two operators commute, and since H_n has odd dimension, by corollary 2.3.3.1, L_1 and L_2 must have a common eigenvector in H_n, i.e. There exists some $B \in H_n \neq 0$, such that:

(2.51) $$L_1(B) = \lambda_1 B \quad \text{and} \quad L_2(B) = \lambda_2 B$$

This implies that any nonzero column of B is an eigenvector of $A \in \mathbb{C}^n$ when n is odd.

ii Inductive step:

Assume any pair of commuting linear operators on a vector space of dimension $n \geq 1$ where the highest power of 2 dividing n is less than 2^k:

$$n = 2^l n', \quad n' \text{odd and } l < k$$

has a common eigenvector.

Now take a space W of dimension n such that:

(2.52) $$n = 2^k n' \quad ,2 \nmid n'$$

We want to show that any complex matrix $A_{n \times n}$ has an eigenvector in $\mathbb{C}$.

Consider the space of $n \times n$ complex symmetric matrices,

(2.53) $$SymM_n(\mathbb{C}) = \{M \in M_n(\mathbb{C}) | M^t = M\}$$

which is a complex vector space with dimension $\frac{n(n+1)}{2}$. Since the highest power of 2 dividing $\frac{n(n+1)}{2}$ is 2^{k-1}, any pair of commuting linear operators on $SymM_n(\mathbb{C})$ has a common eigenvector.

Define L_1, L_2: $SymM_n(\mathbb{C}) \to SymM_n(\mathbb{C})$ by

(2.54) $$L_1(B) = AB + BA^T \quad \text{and} \quad L_2(B) = ABA^T$$

which commute.

Note that $L_1(B)$ and $L_2(B)$ are elements of $SymM_n$ since:

$$AB + BA^T = AB + B^T A^T = AB + (AB)^T$$

which is symmetric, and

$$(ABA^T)^T = (A^T)^T B^T A^T = ABA^T$$

Since they commute and have dimension $\frac{n(n+1)}{2}$, L_1 and L_2 have a common eigenvector, i.e, some nonzero $B \in SymM_n(\mathbb{C})$ satisfies

(2.55) $$AB + BA^T = \lambda B \quad \text{and} \quad ABA^T = \mu B$$

where $\lambda, \mu \in \mathbb{C}$.

Applying A to the first equation on the left and simplifying, we obtain

(2.56) $$A^2B + ABA^T = A\lambda B$$

(2.57) $$A^2B + \mu B = \lambda AB$$

Thus

(2.58) $$(A^2 - \lambda A + \mu)B = 0$$

Using the quadratic formula in the following way:
Since

$$x^2 - \lambda x + \mu = 0$$

and

$$(x - \alpha)(x - \beta)$$

Thus

$$A^2 - \beta A - \alpha A + \alpha\beta I$$

$$A^2 - (\alpha + \beta)A + \alpha\beta I$$

$$\alpha + \beta = \lambda$$

$$\alpha\beta = \mu$$

we may rewrite 2.58 as

$$(A - \alpha I)(A - \beta I)B = 0 \tag{2.59}$$

for some $\alpha, \beta \in \mathbb{C}$

There are thus, four cases:

Note: the product of two non-zero square matrices is zero, then both factors must be singular.

1) $(A - \alpha I)$ is singular, then α is an eigenvalue for A.
2) $(A - \beta I)$ is singular, then β is an eigenvalue for A.
3) $(A - \alpha I)$ is not singular, then $(A - \beta I)B = 0$ which implies that any column vector of B is an eigenvector of A with eigenvalue β.
4) $(A - \beta I)$ is not singular, then $(A - \alpha I)B = 0$ which implies that any column vector of B is an eigenvector of A with eigenvalue α.

This concludes the proof of theorem 2.3.1.

■

2.4 Complex Analysis: Mean Value Property proof

The idea for this elegant proof is presented in a publication of the University of Queensland by Rudolf Výborný [4].

Remark. *The Mean Value Property for harmonic functions on an open ball B_a^R in $\mathbb{C}$*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + te^{i\phi})d\phi \quad \text{for any} \quad 0 < t < R \tag{2.60}$$

Remark. *The ML inequality for complex valued functions $f(z)$ on a contour Γ states that if $|f(z)|$ bounded in Γ, then*

$$\left| \int_\Gamma f(z)dz \right| \leq ML \tag{2.61}$$

Where M is the maximum bound of $|hnkf(z)|$ in Γ and $L = l(\Gamma)$ which is the arc length of Γ

Theorem 2.4.1. *The Fundamental Theorem of Algebra*
Every polynomial of degree $n \geq 1$ has a root in $\mathbb{C}$.

Proof. Suppose there exists a polynomial

$$f(z) = a_n z^n + \dots + a_1 z + a_0 \quad n \geq 1 \quad a_i \in \mathbb{C} \quad a_0 \neq 0 \tag{2.62}$$

which has no roots in $\mathbb{C}$.

We may define a new function

$$g(z) = \frac{1}{f(z)} \tag{2.63}$$

which is also continuous because of the hypothesis.

Since

$$\lim_{x \to \infty} \frac{1}{f(z)} = 0 \tag{2.64}$$

$\exists\, r > 0$ such that:

$$\frac{1}{|f(z)|} < \frac{1}{2|f(0)|} \quad for \quad |z| > r \tag{2.65}$$

Using the *M.V.P.* on $z = 0$ with $t > r$ we obtain

$$\frac{1}{f(0)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{f(te^{i\phi})} \tag{2.66}$$

Then, applying the ML inequality on 2.65, with $L = 2\pi$ and the maximum $M = \dfrac{1}{2f(0)}$ (since $t > r$), we obtain

$$\frac{1}{|f(0)|} \leq \frac{1}{2\pi} \frac{1}{2|f(0)|} 2\pi = \frac{1}{2|f(0)|} \tag{2.67}$$

Which yields a contradiction, thus proving theorem 2.4.1.

■

2.5 Complex Analysis: Liouville's Theorem proof

A proof similar to this one is presented in the last subsection of *Proofs of the Fundamental Theorem of Algebra* by M atthew Steed [5].

We shall use Liouville's Theorem stated in the following manner:

Remark. *A bounded entire function is constant.*

Note that *entire* means that $f(z)$ is analytic at all finite points of $\mathbb{C}$.

Theorem 2.5.1. *A polynomial of $degree \geq 1$ in $\mathbb{C}$ has n complex roots with possible multiplicity.*

Proof. Given a polynomial

(2.68) $$f(z) = a_n z^n + \ldots + a_1 z + a_0, \quad a_i \in \mathbb{C}, \quad a_0 \neq 0$$

Suppose the function $|f(z)|$ has a minimum at some point $z = a$ and $|f(a)| \neq 0$.

Define the inverse function

(2.69) $$g(z) = \frac{1}{|f(z)|}$$

for which $\frac{1}{|f(a)|}$ is now a maximum.

Note that $\frac{1}{|f(z)|}$ is holomorphic on all $\mathbb{C}$ and it is also bounded since $\lim_{z\to\infty} \frac{1}{|f(z)|} = 0$.

By Liouville's Theorem, the function 2.69 must be constant, which would mean that $|f(z)|$ is constant, i.e. $n = 0$, which yields a contradiction. Thus proving theorem 2.5.1,

■

Other properties of Polynomials

The outlines of the first two proofs are also found in *The Fundamental Theorem of Algebra* [1].

3.1 Factorization

Lemma 3.1.1.

$$\frac{z^n - w^n}{a - b} = \sum_{k=0}^{n-1} z^k w^{n-1-k} \tag{3.1}$$

Proof. We will use induction to prove lemma

i Basis step

$$\frac{z-w}{z-w} = \sum_{k=0}^{0} z^k w^{n-1-k} = z^0 w^0 = 1 \tag{3.2}$$

ii Inductive Step:

Assume

$$\frac{z^n - w^n}{z - w} = \sum_{k=0}^{n-1} z^k w^{n-1-k} \tag{3.3}$$

Multiplying both sides of the equation 3.3 by w we get:

$$\frac{w(z^n - w^n)}{z - w} = w \sum_{k=0}^{n-1} z^k w^{n-1-k} \tag{3.4}$$

On the other hand, multiplying both sides of the equation 3.3 by z^n we obtain:

$$\frac{z^n(z-w)}{z-w} = z^n \tag{3.5}$$

Adding together equations 3.4 and 3.5 yields the result:

$$\frac{w(z^n - w^n) + z^n(z-w)}{z-w} = w \sum_{k=0}^{n-1} z^k w^{n-1-k} + z^n \tag{3.6}$$

$$\frac{wz^n - w^{n+1} + z^{n+1} - wz^n}{z-w} = w \sum_{k=0}^{n-1} z^k w^{n-1-k} + z^n$$

$$\frac{z^{n+1} - w^{n+1}}{z-w} = \sum_{k=0}^{n-1} z^k w^{n-k} + z^n$$

Thus

$$\frac{z^{n+1} - w^{n+1}}{z - w} = \sum_{k=0}^{n} z^k w^{n-k} \tag{3.7}$$

■

Theorem 3.1.2. *Given a polynomial*

$$f(z) = a_n z^n + ... + a_1 z + a_0 \quad n \geq 1 \quad a_i \in \mathbb{C} \quad a_n \neq 0 \quad a_0 \neq 0 \tag{3.8}$$

for any complex number $w \in \mathbb{C}$ *, we have* $f(w) = 0 \iff$ *there exists another polynomial* $g(z)$ *of degree* $n-1$ *such that*

$$f(z) = g(z)(z - w) \quad \forall z \in \mathbb{C} \tag{3.9}$$

Proof. ($\Longrightarrow$)

Suppose that $f(w) = 0$, then we have

$$f(w) a_n w^n + ... + a_1 w + a_0 = 0 \tag{3.10}$$

Thus

$$f(z) = f(z) - f(w) \tag{3.11}$$

$$f(z) = a_n(z^n - w^n) + a_{n-1}(z^{n-1} - w^{n-1}) + ... + a_1(z - w)$$

Using lemma 3.1.1:

$$f(z) = a_n(z - w) \sum_{k=0}^{n-1} z^k w^{n-1-k} + a_{n-1}(z - w) \sum_{k=0}^{n-2} z^k w^{n-2-k} + ... + a_1(z - w) \tag{3.12}$$

$$f(z) = (z - w) \sum_{m=1}^{n} \left(a_m \sum_{k=0}^{m-1} z^k w^{m-k} \right)$$

We may take

$$g(z) = \sum_{m=1}^{n} \left(a_m \sum_{k=0}^{m-1} z^k w^{m-k} \right) \forall z \in \mathbb{C} \tag{3.13}$$

Thus $f(z)$ can be written as

$$f(z) = (z - w) g(z) \tag{3.14}$$

where $g(z)$ is a degree $n-1$ polynomial.

($\Longleftarrow$)

Suppose there exists an $n-1$ degree polynomial such that

$$f(z) = (z-w)g(z) \tag{3.15}$$

I is trivial that

$$f(w) = (w-w)g(x) = 0 \tag{3.16}$$

■

3.2 Maximum number of roots

Theorem 3.2.1. *There are at most n distinct complex numbers w for which $f(w) = 0$, i.e. f has at most n distinct roots.*

This is equivalent to f being factored into at most n linear factors.

Proof. Using once again induction, but this time over the degree n of the polynomial:

i Basis step

$$f(z) = a_1 z + a_0 \tag{3.17}$$

which is already linear and has only one root, $\frac{-a_0}{a_1}$

ii Inductive Step:

Assume all polynomials $g(z)$ of degree $n-1$

$$g(z) = a_{n-1}z^{n-1} + ... + a_1 z + a_0 \tag{3.18}$$

can be factored into at most $n-1$ linear factors.

Then $f(z)$ with degree n can be written as follows by theorem 3.1.2

$$f(z) = g(z)(z-w) \quad \text{for some } g(z) \tag{3.19}$$

Since $g(z)$ can be factored into at most $n-1$ factors, $f(z)$ can be factored into n linear factors.

■

3.3 Complex Conjugate Root Theorem

Theorem 3.3.1. *Given a polynomial $f(z)$ with real coefficients with $z_0 = a + bi$ as a root, then the complex conjucate $\bar{z_0} = a - bi$ is also a root of $f(z)$.*

Proof. We have that

$$f(z_0) = a_n z_0^n + a_{n-1} z_0^{n-1} + \ldots + a_0 = 0 \tag{3.20}$$

In summation notation:

$$f(z_0) = \sum_{j=0}^{n} a_j z_0^j = 0 \tag{3.21}$$

Using the properties of the complex conjugate:

$$f(\overline{z_0}) = \sum_{j=0}^{n} a_j \left(\overline{z_0^j}\right) = \sum_{j=0}^{n} \left(\overline{a_j z_0^j}\right) = \left(\overline{\sum_{j=0}^{n} a_j z_0^j}\right) \tag{3.22}$$

Since

$$\overline{0} = 0 \tag{3.23}$$

thus,

$$\overline{\sum_{j=0}^{n} a_j z_0^j} = \overline{0} = 0 \tag{3.24}$$

■

Conclusions

The Fundamental Theorem of algebra is at the backbone of several, very different branches of mathematics. Although the theorem has been proved before, it can be used to show the flexibility and applicability of mathematics as a whole. As for the proofs themselves, it is easy to see that they become easier and smaller as the mathematics behind them gets more complex.

In regard of further work,since most of the proofs share the common denominator, which is the use of properties shared by all continuous (and sometimes also bounded) functions, one can use the proofs and ideas presented in this research paper to further develop proofs for different kinds of continuous functions.

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