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Introducción a la Topología Fractal

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Introducción A La Topología Fractal

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Índice general

Resumen

En este texto se intentará hacer una introducción a la topología y geometría fractal. Para esto, va a ser necesario definir algunas bases en topología y análisis que permitan expandir el conocimiento básico presentado a nivel universitario, de manera que la idea de lo que es un fractal se forme a partir de las bases presentadas. Este texto es una guía rápida para las personas que tengan algún conocimiento formal en matemáticas y puede servir como impulso para aquellos que busquen aprender mas sobre fractales o topología.

UNIVERSIDAD SAN FRANCISCO DE QUITO

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Colegio de Ciencias e Ingenierías

An Overview of Fractal Topology

Francisco José Iturralde

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Abstract

In this text I will attempt to set the basic ideas of the topology and geometry of fractals. For this solid grounds on topology and analysis should be settled first, in a way that the idea of fractals build up from more classical areas of mathematics. This text has also a quick guide for the people that have already little formal mathematical basis, so that they can catch up on what is needed to begin studying the mathematical composition of fractals.

Chapter 0

Introduction

The notion that nature can't be fully described by the traditional empirical models is a premise of the theory of the study of the twin subjects of dynamical systems (chaos) and fractal geometry. And it is a profound yet clear thought, since it declares that our study of nature itself is confined by the precision and deepness of our knowledge. Hence any attempt to go deeper into our description of nature is either part of a refinement of our knowledge or exploration of new horizons in mathematics, philosophy, science or engineering.

From this point onward, more mathematical terms will be used since we will be talking about more specific topics, so that until the end of this chapter we will talk about the settlements of fractal geometry.

0.1. Spaces

To clarify any type of misunderstanding, some background on topological and metric spaces should be defined for the development of the text. We will not delve inside the abstract notions of spaces, so we will just define *space* as an ordered pair (X, s), where X is any set, and s is a structure defined on the set, that can be from an algebraic structure, a topology, a norm, a metric, a measure, among many others; but for this text we may consider topological and metric spaces, along with other occasional mentions.

Topological Spaces

The first structure we will consider for the spaces here is the topology, further ahead it will come clear how a topological space is the deepest level of abstraction where fractals can be defined; so for now we should focus on defining what topological spaces are, so:

Definition 0.1.1 (Topological Space)

A topological space (X, τ) is an ordered pair, where X is a set and τ is a collection of subsets in X with the following properties:

- $\emptyset \in \tau$
- $\bigcup\limits_{\alpha \in A} V_\alpha \in \tau$
- $V_1 \cap V_2 \in \tau$

Where A is an arbitrary index family. And every $V_i \in \tau$.

Topological spaces should be our first structure to understand fractals, and will provide some relevant properties that will become clearer as we progress. So, we got our basic structure, and we can make a lot of thought experiments on how this will work for our objectives, but we should refine our structure a little bit more before settling in with a intuitive and comfortable space to work with.

Metric Spaces

It is a well known fact that metric spaces solve and guide us in many of our notions of modern mathematics, and in fractal topology it is no exception. The idea of defining a distance in a topological space gives us a more intuitive idea on how "close"points may be one from another. So, as we did before, we should define it first:

Definition 0.1.2 (Metric Space)

Let $\mathbb{R}^+$ be the set of positive real numbers.

A metric space is an ordered space (X, d), where X is any set and d is a metric defined on that set.

The metric should satisfy the following:

- $d : X \times X \to \mathbb{R}^+ \cup \{0\}$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)\ \forall x, y \in X$
- Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$

In further chapters we will talk more about metric spaces, but another preliminary definition should be made in order to finish the groundwork needed for fractals.

Definition 0.1.3 (Equivalence of Metrics)

Let X be a metric space, and let d_1 and d_2 be metrics defined of that space.

We say that those metric are equivalent if there exist numbers $r_1, r_2 \in \mathbb{R}^+$, $r_1 < r_2$, such that:

$$r_1 \cdot d_2(x, y) \leq d_1(x, y) \leq r_2 \cdot d_2(x, y) \qquad \forall x, y \in X$$

We denote this equivalence in X as $d_1 \sim d_2$

Another way to see this is that two metrics are equivalent if they induce the same topology on the space. Clearly this helps us find an equivalence relation between metrics, and when we work with fractal spaces we should be able to find some useful applications. If $r = r_1$ or $r = r_2$, and $d_1(x, y) = r \cdot d_2(x, y)$, we say that d_1 is a scaling of d_2. And we call r a scaling factor.

So we have shown that metrics can be equivalent, but for metric spaces to be equivalent we should also take some considerations for the spaces, so without delving deep in real analysis topics, we can say that:

Definition 0.1.4 (Equivalence of Metric Spaces)

Let (X_1, d_1) and (X_2, d_2) be metric spaces, then let h be a bijective function from X_1 to X_2.

Let $\hat{d}_1 \sim d_1$ for some metric $\hat{d}_1$ in X_1, such that:

$$\hat{d}_1(x, y) = d_2(h(x), h(y)) \qquad \forall x, y \in X_1$$

Then we say the spaces are equivalent, denoted as: $(X_1, d_1) \sim (X_2, d_2)$ or $X_1 \sim X_2$.

We can see that any two equivalent metric spaces are homeomorphic, but the converse may not hold. This equivalence of metrics and metric spaces will comes handy when we are looking at any type of deformation of spaces.

A useful tool we will be using a lot in the theory of fractals is the dilation, and in metric spaces it is a part of the *affine group*, defined as follows:

Definition 0.1.5 (Affine Group)

Let $(\mathbb{R}^n$, d) be a metric space.

Let $A \in GL_n(\mathbb{R})$ and $v \in \mathbb{R}^n$.

Then, the *Affine Group of transformations in* $\mathbb{R}$ is defined by:

$$\text{Aff}_n(\mathbb{R}) = \left\{ \left(\begin{array}{cc} A & v \\ 0 & 1 \end{array} \right) \middle| A \in GL_n(\mathbb{R}), v \in \mathbb{R}^n \right\}$$

If $det(A) > 0$ we say that the transformation is orientation-preserving.

If $det(A) < 0$ we say that the transformation is orientation-reversing.

Some texts describe the affine group as the subgroup of $GL_{n+1}(\mathbb{R})$ that acts on $\mathbb{R}^n \times \{1\} \simeq \mathbb{R}^n$ that preserves collinearity and ratios of distances [6]. It is useful to think of the affine group of transformations as a combination of several actions on $\mathbb{R}^n$, namely scalings, rotations, reflections, translations, shears and composition of them. [7]

Another way to see an affine transformation is through a function $F : \mathbb{R}^n \to \mathbb{R}^n$:

$$F(x) = Ax + b \; ; \; A \in GL_n(\mathbb{R}), b \in \mathbb{R}^n \tag{0.1}$$

The Affine group is a useful yet deep tool we will be using for several areas in this text, but we will stick to this definition and we will take some subgroups like the *Isometry Group of transformations in* $\mathbb{R}$ and concepts like conformal transformations for further applications.[6] We will see clearly how everything becomes more useful for our purpose, for now we can settle with this as our basic tools for the study of spaces.

Complete, Compact and Connected Spaces

For this text, several basic ideas will be overlooked, for example, the notions of convergent sequences, Cauchy sequences and limit points but we will be using these notions to understand compact, complete and connected spaces.

Definition 0.1.6 (Complete Spaces)

A metric space $(X,\ d)$ is complete provided that every Cauchy sequence converges.

Furthermore, we can say that a space (X, d) is *closed* space contains all of its limit points and the space is *bounded*, if there exists an open ball B_r^k, with $r \in \mathbb{R}$, such that $X \subseteq B_r^k$.

We define a *cover* of a space (X, d) as a collection of open sets $\{A_\alpha\}$ with $\alpha \in \Lambda$ and Λ as a set of indexes[1], such that $X \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha$; then we call a space (X, d) *compact*, provided that every cover of X has a finite *subcover*, that means a finite subset of $\{A_\alpha\}$ covers X.

Theorem 0.1.1 (Heine–Borel theorem)

Let $X \subset \mathbb{R}^n$ and d as the usual metric, and (X, d) be a complete metric subspace.

Let $S \subseteq X$. Then S is compact if and only if it is closed and bounded.

The proof of this theorem follows from the general knowledge of point-set topology and can be found in most topology books [4], there you can find as well that the Heine-Borel Theorem holds as long as $X \subseteq \mathbb{R}^n$ and the metric d is equivalent to the euclidean metric. Finally, to settle the foundations needed for topology, we need to talk about connected sets, we will go directly to the definition

[1] Λ can be uncountably infinite.

in order to begin working with fractals promptly.

Definition 0.1.7 (Connected Space)

A metric space (X, d) is connected provided the only clopen[a] subsets are the empty set and X. Similarly a subset $M \subseteq X$ is connected provided it is connected with the induced topology/metric.

[a]Clopen: Simultaneously open and closed

We will understand disconnected as any space that does not meet the previous definition. and totally disconnected as a space where the singletons or points are the only connected subsets. [1]

0.2. Fractals

The book "The Fractal Geometry of Nature"by Benoît Mandelbrot [3] declared many intricacies about the structure of nature and how does it work and served as inspiration to several authors to deepen the knowledge of fractals, developing its uses in physics, computation, chemistry, biology and several other areas. In order to develop an understanding of what fractals are, there are many approach that can be taken, from dimension theory to abstract algebra, algebraic topology or studying symmetries along with scale invariant properties. That the idea of self-similarity as a scale invariant property, and from that perspective figure out deeper studies of the properties of the space where fractals live. Hence a multidisciplinary work will be very useful in the study of fractals.

Fractals represent themselves mostly in irregular and unpredictable ways, but the study of fractals becomes more clear when it begins with the more regular ones and continues developing from there. That's why the most famous fractals are regular, because their structure is more easily described and they can be modeled with more ease. Things like the Sierpinski triangle, Cantor's dust and Koch curve (or snowflake) are what we call regular fractals and they can be described by a sequence of infinite iterations that create a self-similar structure, hence it is appropriate to introduce our first definition.

Definition 0.2.1 (Fractal)

A Fractal $\mathfrak{F}$ is a complete metric space that is self-similar at any scale.

For now we will dabble with this definition, because in further chapters, we will explore what is and how self-similarity is manifested in fractals. This will include how dimension remains constant along a change of scale and some other properties that remain invariant following self-similarity.

Hausdorff sets and spaces

Following Barnsley[1] introduction to fractal geometry and the previously notions of metric spaces and fractals, We need to begin working with a deeper level of space that allow somehow the idea of self similarity to develop. So we may find quite useful to work with the compact subsets of a space as our main focus in contrast with the open and closed sets.

Definition 0.2.2 ($\mathcal{H}(\bullet)$ / Hausdorff Set)

Let $(X,\, d)$ be a complete metric space, and let $\mathcal{H}(X)$ be the set whose elements are all the compact subsets of X different from the empty set.

The set $\mathcal{H}(X)$ is called the *Hausdorff set of* X.

Now, the proper thing to do is to define a metric for the Hausdorff sets, we consider the usual set distance as an initial reference metric for them to be complete metric spaces. The notion of metric may seem couterintuitive with this change, but will make sense as we move forward. So, the set distance from a subset into another is constructed upon the idea of a the set distance from a point to a set, in Hausdorff spaces, they are both considered points, but we may need some additional considerations.

So, the minimum and maximum set distances[2] from a singleton point $\{p\} \in \mathcal{H}(X)$ to a compact set $Q \in \mathcal{H}(X)$ is defined as follows:

$$\underline{d}(p, Q) = \text{ínf}\,\{d(p, q) | q \in Q\} \tag{0.2}$$

$$\overline{d}(p, Q) = \sup\,\{d(p, q) | q \in Q\} \tag{0.3}$$

Furthermore, we can introduce the distance from a non-singleton compact set P to another Q. We will use the equation 0.2 as a reference, such that:

$$D(P, Q) = \sup\,\{\underline{d}(p, Q) | p \in P\} \tag{0.4}$$

Or, equivalently, using equation 0.2

$$D\,(P, Q) = \sup\,\{\text{ínf}\,\{d(p, q) | q \in Q\} | p \in P\} \tag{0.5}$$

It is important to remark that, even though this may seem as a metric in $\mathcal{H}(X)$, it lacks the commutative property, so we don't always have the case that $D(P, Q) \neq D(Q, P)$, as seen on the figure below:

[2]We usually call set distance the one we define as minimum set distance.

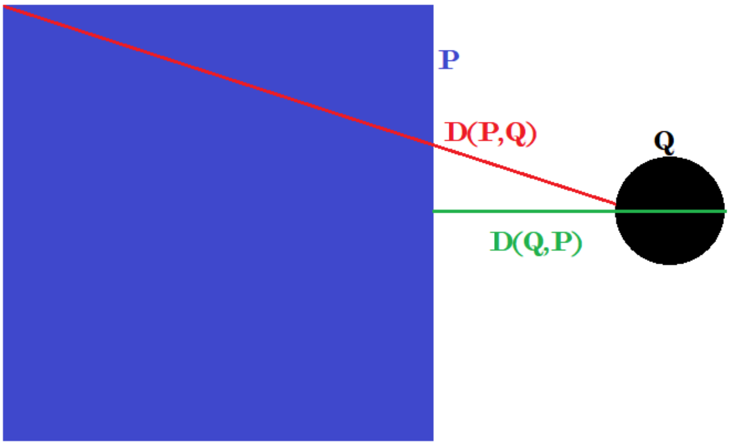

Figure 0.1: Lengths representing the distances $D(P,Q)$ and $D(Q,P)$

This shows us that we will need some considerations in order to define a metric in $\mathcal{H}(X)$. For instance, final consideration for this chapter is what we will use for the metric of sets in $\mathcal{H}(X)$, which is called the Hausdorff metric.

$$h(P,Q) = \sup\left\{D(P,Q), D(Q,P) | P, Q \in \mathcal{H}(X)\right\} \tag{0.6}$$

With this we claim the following:

Proposition 1

h is a metric for $\mathcal{H}(X)$

Proof First, let $A, B, C \in \mathcal{H}(X)$.

Now we can clearly see that $h(A,B) \geq 0$, since it considers the supremum and infimum of a set of numbers in $\mathbb{R}^+ \cup \{0\}$. Furthermore $h(A,A) = D(A,A) = 0$ which shows that $h(A,A) = [\sup_{a \in A} [\text{ínf}_{\hat{a} \in A} d(a,\hat{a})]] = [\sup_{a \in a} 0] = 0$.

Also, it is easy to see that:

$$\sup\{D(A,B), D(B,A)\} = \sup\{D(B,A), D(A,B)\}$$

Therefore,

$$h(A,B) = h(B,A)$$

Perhaps a more tricky part to see is the triangle inequality, but developing from the definition we get:

$$\begin{aligned}
h(A,B) &= \sup\left\{\sup_{a\in A}[\operatorname{ínf}_{b\in B} d(a,b)], \sup_{b\in B}[\operatorname{ínf}_{a\in A} d(a,b)]\right\} \\
&\leq \sup\left\{\sup_{a\in A}[\operatorname{ínf}_{b\in B}[d(a,c)+d(c,b)]], \sup_{b\in B}[\operatorname{ínf}_{a\in A}[d(a,c)+d(c,b)]]\right\} \forall c\in C \\
&= \sup\left\{\sup_{a\in A}\left[d(a,c)+\operatorname{ínf}_{b\in B}[d(c,b)]\right], \sup_{b\in B}\left[d(c,b)+\operatorname{ínf}_{a\in A}[d(a,c)]\right]\right\} \forall c\in C \\
&= \sup\left\{\sup_{a\in A}[d(a,c)]+\operatorname{ínf}_{b\in B}[d(c,b)], \sup_{b\in B}[d(c,b)]+\operatorname{ínf}_{a\in A}[d(a,c)]\right\} \forall c\in C \\
&\leq \sup\left\{\sup_{a\in A}[d(a,c)]+\sup_{c\in C}\left[\operatorname{ínf}_{b\in B}[d(c,b)]\right], \sup_{b\in B}[d(c,b)]+\sup_{c\in C}\left[\operatorname{ínf}_{a\in A}[d(a,c)]\right]\right\} \\
&\leq \sup\left\{\sup_{a\in A}[d(a,c)|\text{ c fixed}]+D(C,B), \sup_{b\in B}[d(b,c)|\text{ c fixed}]+D(C,A)\right\} \\
&\leq \sup\left\{\sup_{a\in A}[\operatorname{ínf}_{c\in C} d(a,c)], \sup_{c\in C}[\operatorname{ínf}_{a\in A} d(a,c)]\right\} + \sup\left\{\sup_{c\in C}[\operatorname{ínf}_{b\in B} d(c,b)], \sup_{b\in B}[\operatorname{ínf}_{c\in C} d(c,b)]\right\} \\
&= h(A,C) + h(C,B)
\end{aligned}$$

With this, the triangle inequality is proven, and the proposition of h being a metric for $\mathcal{H}(X)$. ■

This provides the final tool needed to define the appropriate working space onward.

Theorem 0.2.1

Let (X,d) be a complete metric space and let $\mathcal{H}(X)$ be its Hausdorff set, with h as a metric.

Then $(\mathcal{H}(X),h)$ is a complete metric space.

We call it *Fractal space.*

Proof Let $\{Y_n\}$ with $Y_n \in \mathcal{H}(X)$ be a Cauchy sequence with points in $(\mathcal{H}(X), h)$, then there exists a Cauchy sequence $\{y_n^i\}$, with $y_n^i \in Y_n \subseteq X$, such that as $i, n \to \infty$, $y_n^i \to y$, we have:

$$d(y_n^i, y) \to 0$$

Furthermore if we only take the index $i \to \infty$, $y_n^i \to y_n$, gives $d(y_n^i, y_n) \to 0$, which makes the point y_n the point in Y_n that minimizes the distance $d(a_n, y)$, where a_n is any point in Y_n, i.e.:

$$d(y_n, y) = \text{ínf}\{d(a_n, y) | a_n \in Y_n\}$$

This shows us that if we fix i, we find a convergent sequence using points in the compact sets $\{Y_n\}$; and if we fix n, the indexes $i = 1, 2, 3 \ldots$ describe a convergent sequence $\{y_n^i\}$ in the compact set Y_n.

Now we can find that $\{Y_n\}$ is bounded, since:

$$h(Y_n, Y_m) \leq \sup\{d(y_n^i, \hat{y}_m^i); y_n^i \in Y_n, \hat{y}_m^i \in Y_m\} \qquad n, m, i \in \mathbb{N} \tag{0.7}$$

Since $\{y_n^i\}$ is a convergent sequence in X, then we can say that:

$$y^i = \lim_{n \to \infty} y_n^i \tag{0.8}$$

We now define a set $Y \in \mathcal{H}(X)$ as follows:

$$Y = \left\{ y^i \in X \middle| y_n^i \in Y_n \text{ converges to } y^i \right\} \tag{0.9}$$

The sequence $\{y_n^i\}$ converges to y^i by our initial preposition, so the sequence $\{Y_n\}$ converges to Y by the convergence of the sets of points inside each Y_n.

This ends the proof. ■

One way to see the consequences of this theorem is that, in a complete metric space, a Cauchy sequence of compact sets converges into another compact set.

Chapter 1

Dimension

Index terms— Dimension: Euclidean Dimension (D_E), Topological Dimension(D_T); Fractal; Fractal dimension(s): Similarity Dimension (D_S), Hausdorff-Besicovitch Dimension (D_{HB})

1.1. Introduction

This chapter will focus on dimension, especially the discussion that concerns the study of fractals. To do that a concise definition of dimension should be made and it should lay some ground for the foundations on fractal dimension. I will take for granted that the reader is familiar with the notions of metric spaces and has a basic idea of topology. Nevertheless, I will explain some details if I consider them necessary.

Roughly speaking, dimension is the number of independent components in a space to move around. In general we can define what dimension is for an euclidean space or manifold in a straightforward way; but we won't talk about the inductive construction of the definition of dimension. The development of dimension theory can be found in the referenced literature. [2]

For an arbitrary complete metric space, the definition may need more rigorous specifications. Hence, the first thing that should be done is to define dimension for a space X, denoted as $dim(X)$, we will see that this is not the only approach to define a dimension, but it is appropriate to begin from here.

Definition 1.1.1 (Dimension)

The dimension of an Euclidean space, $\mathbb{R}^n$, is the number of independent coordinates needed to represent it $dim(\mathbb{R}^n) = n$.

The dimension of an open ball in $\mathbb{R}^n$ of radius r, centered at p is the dimension of the space it is represented in: $dim(B_r^n(p)) = dim(\mathbb{R}^n) = n$.

Let (X, d) be a complete metric space.

Let $p \in X$, let $\epsilon > 0$, we denote $dim_p(X)$ as the dimension of an open neighborhood around p, $U_p \subset B_\epsilon^n$. Such that there exists a homeomorphism $F : U_p \to V \subset B^k$, as $\epsilon \to 0$. That is the minimum arbitrarily small set homeomorphic to a k-ball that V can be contained in, such as in figure 1.1.

$$dim_p(X) = \lim_{\epsilon \to 0} dim(U_p) = dim(V) = dim(B^k) = k \tag{1.1}$$

The dimension of the space (X, d) is defined by:

$$dim(X) = sup\{dim_p(X)|p \in X\} \tag{1.2}$$

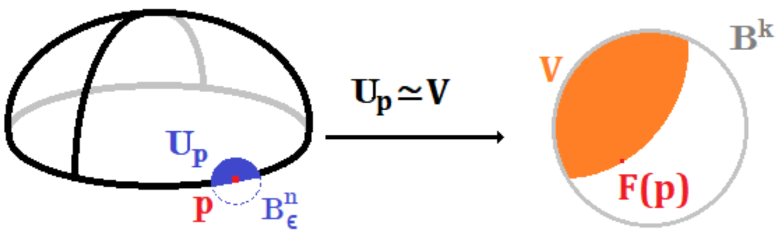

Figure 1.1: Homeomorphism between a neighborhood U_p and $V \subseteq B_\epsilon^k$

Even though we will not use this definitions of dimension throughoutly, we will use them as a reference of the classical notion of dimension and we will build from it onwards.

Something that should be important to remark is that it suffices that a subset of U_p to be homeomorphic to an k-ball to have dimension k.

Note that this aligns with the definition of dimension when we talk about manifolds, but to generalize it to the appropriate level we may need a couple of extra definitions. We should stop here with the definition of dimension, since we have reached an appropriate level of abstraction and going further will imply going beyond the scope of this paper.

Euclidean Dimension

Recall that in linear algebra we consider vector spaces over fields, $\mathbb{R}$ (In this will use $\mathbb{R}$, or $\mathbb{C}$ when mentioned)[1], the classic example being $\mathbb{R}^n$, these spaces also have a usual metric and topology, which we will be using onward. We then use the number of coordinates of $\mathbb{R}$ to define the dimension of the space, that means the minimum number of vectors needed to generate $\mathbb{R}^n$, this defines an euclidean space and an euclidean dimension as follows:

Definition 1.1.2 (Euclidean Dimension)

Let (X, d) be a complete metric space, and $X \subseteq \mathbb{R}^n$.

The euclidean dimension of X is defined as:

$$D_E(X) = dim(\mathbb{R}^n) = n$$

We say that n is the euclidean dimension of X, and we call $D_E(U)$ the Euclidean dimension of a subset $U \subseteq X \subseteq \mathbb{R}^n$. This can be seen as the dimension of the ambient space we are working in or as the dimension of the space that contains the sets X and its subsets. Even though this is not the only definition of dimension

[1]Considering that $\mathbb{C} \cong \mathbb{R}^2$

we will be working, it will be restraining the grounds for other definitions and approaches to other dimensions.

Topological Dimension for a Manifold

Recall that a k-manifold is defined as follows:

Definition 1.1.3

A k-manifold $\mathcal{M}$ in $\mathbb{R}^n$, is a subspace of $\mathbb{R}^n$ that is locally homeomorphic to $\mathbb{R}^k$, for $n \geq k \in \mathbb{N}$

We will consider manifolds with boundary further in the text, but what we are interested in right now is the dimension of the manifold. For that we need to define a another type of dimension.

Definition 1.1.4 (Topological Dimension of a Manifold)

Let $\mathcal{M}$ be a k-manifold in $\mathbb{R}^n$, then we have:

$$D_T\left(\mathcal{M}\right) = dim_p\left(\mathcal{M}\right) = k$$

$\forall p \in \mathcal{M}$

Observe that $D_T\left(\mathcal{M}\right) \leq D_E\left(\mathcal{M}\right) \rightarrow k \leq n$. Also note that this definitions holds for manifolds with and without boundaries, following the former definition of dimension. To extend this definition to arbitrary subsets of $\mathbb{R}^n$, we can take the classical definition of cover of topology and introduce the refinements of covers of compact sets. [8]

Definition 1.1.5 (Refinement of a Cover)

Let $\{\{A_l\}_m\}$ be the set that contains the open covers of $S \in \mathcal{H}(X)$, with indexed families $l \in \Lambda_m$, $m \in M$.

A refinement of an open cover $\{A_i\}_j$ is an open cover $\{A_{i'}\}_{j'}$, with $j, j' \in M$, such that every $A_{i'}$ with $i' \in \Lambda_{j'}$, follows:

$$A_{i'} \subseteq A_i \text{ for some } i \in \Lambda_j$$

In particular, this definition implies that:

$$\bigcup_{i' \in \Lambda_{j'}} A_{i'} \subset \bigcup_{i \in \Lambda_j} A_i$$

In addition to the previous definition we should include that the order of a cover. So we can say that the order of a cover $\{A_i\}$ of a complete metric space X is an integer $m > 0$; such that, for every point $p \in X$, there exists a subset $\{A_{i_j}\} \subseteq \{A_i\}$ that follows, $p \in A_{i_j}$, $\forall j \in \{1, 2, \cdots n\}; n \in \{1, 2, \cdots m\}$. In other words the order of a cover is the maximum number intersecting sets in any point of X. With this the following remark can be made:

Definition 1.1.6 (Topological Dimension of a Compact Set)

The topological dimension for a set $S \in \mathcal{H}(X)$ is an integer $k \geq 0$, such that the minimum refinement of every cover has order $k + 1$. Denoted as $D_T(S) = k$.

The idea of a refinement vaguely introduces the concept of a measure for fractals, said measure will be introduced soon. But for now, perhaps a better way to see this is through examples, consider the Cantor set $\mathfrak{C}$, which is totally disconnected, but it is compact and every point has uncountable many points in its neighborhood. So we can cover the whole set with an open set and recursively find refinements of disconnected sets that cover the first and third third of the cantor set. Hence every cover can be refined in a cover that contains the first and third part of each arbitrarily small iteration, as seen in the figure below.

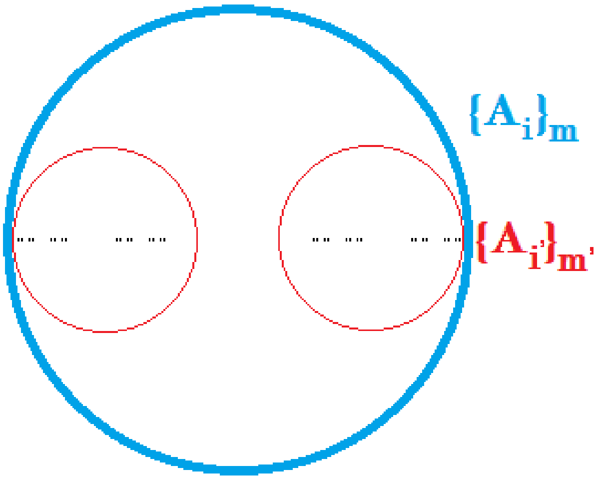

Figure 1.2: Cover Refinement of the cantor set $\mathfrak{C}$ in $\mathbb{R}^2$

This leads to $D_T(\mathfrak{C}) = 0$.

Also every cover can be seen as the refinement of another cover of order 0, since every open cover $\{A_\alpha\}$ can be seen as the refinement of the cover of a single element $\{\bigcup_{\alpha\in\Lambda} A_\alpha\}$. With this we can claim:

Proposition 2

Totally disconnected sets have topological dimension equal to 0.

Proof Without loss of generality, let $\{A_i\}$ be any cover of order 0 of a totally disconnected set $S \in \mathcal{H}(X)$, and let $p_i \in S$, $p_i \in A_i$. Then an open cover $\{A'_i, A"_i\}$ can be the refinement of $\{A_i\}$, as follows: $A'_i = A_i/\{p_i\}$ and $A"_i = \{p_i\}$; that means, by the induced topology the set has a basis of singleton points and every point is an open set in S, hence a refinement can be made that contains every single point and every intersection of different elements two different elements of the refined cover is empty.

■

1.2. Fractal Dimensions

Depending on what approach is taken, we can consider several definitions of fractal dimension and use them for various purposes. For instance, we would like to use a certain kind of dimension called similarity dimension D_S on well-behaved fractals, or fractals that retain its structure with scaled exact copies of themselves; on the other hand if we want a fractal defined by a stochastic process we may use the Hausdorff-Besicovitch Dimension (D_{HB}) or other different dimension for counting methods. Each one of them has a different use and we will be described in each specific case.

Similarity Dimension

Recall that an *open interval* in $\mathbb{R}$ is denoted by (a, b), for $a \leq b$; and we call an open set an *open cell or open box in* $\mathbb{R}^n$, $C \in \mathbb{R}^n$ if it is the direct product of open intervals, as follows:

$$
\begin{aligned}
B &= \Pi_{k=1}^{n}(a_k, b_k) | -\infty < a_k \leq b_k < \infty \\
&= (a_1, b_1) \otimes (a_2, b_2) \otimes \cdots (a_n, b_n) \text{ for } a_i \leq b_i, i = 1, 2, \cdots n \\
&= \{(x_1, x_2, \cdots x_n) | a_i < x_i < b_i; i = 1, 2, \cdots n\}
\end{aligned}
$$

Note that $a_i = b_i$ is still an option and the open set $(a_i, a_i) = \emptyset$, as well as the direct product of the empty set with any set is the empty set itself.[2] So, without further a do, we define a couple of terms:

[2]Note as well that open cells form a basis on the topology of $\mathbb{R}^n$.

Definition 1.2.1 (Elementary sets)

Let $\{B_\alpha\}$ be a collection of pairwise disjoint bounded open cells in $\mathbb{R}^n$, for $\alpha \in \Delta$, and Δ as a family of indexes. If Δ is finite, we call $E = \bigcup_{\alpha\in\Delta} B_\alpha$ an *elementary set.*

Let $\{C_\alpha\}$ be the collection of sets, such that $C_\alpha = \overline{B_\alpha}$, means C_α is the closure for each B_α. It is straightforward to see that the closure of an elementary set is the union of the closure of it's disjoint bounded cells, $\overline{E} = \bigcup_{\alpha\in\Delta} C_\alpha$. Since $\overline{E}$ is closed and bounded in some $X \subseteq \mathbb{R}^n$, we can say that $\overline{E} \in \mathcal{H}(X)$.

Definition 1.2.2 (IFS - Iterated Function System)

Let (X, d) be a complete metric space.

An *iterated function system* is an ordered pair $(F, \{w_i\})$, where $f \in \mathcal{H}(X)$ and $\{w_i\}$ is a finite set of contractive[a] affine transformations that go from F to F, such that:

$$w_i(F) \cap w_j(F) \neq w_k(F) \qquad \forall i \neq j; \quad i, j, k = 1, 2, \cdots m \tag{1.3}$$

Each w_i has a *contractivity factor* $s_i = \det(A_i)$, where A_i is the matrix of the affine transformation.

[a] A contractive affine transformation is an affine transformation whose matrix $A \in GL_n(\mathbb{R})$ has $\det(A) < 1$.

Suppose[3] we have a set $\mathfrak{F} \in \mathcal{H}(X)$, where X is a complete metric space within $\mathbb{R}^n$, that means $D_E(X) = D_E(\mathfrak{F}) = n$ and $D_E(F) = k \leq n$. Then if we have a proper subset $\mathfrak{F}_1 \subset \mathfrak{F}$ that is homeomorphic to $\mathfrak{F}$, we get what we call a regular fractal, then for a metric space we can safely define:

[3] We will use fraktur letters to talk about fractals and capital letters to represent topological spaces.

Definition 1.2.3 (Regular Fractal)

Let (X, d) be a complete metric space with $D_E(X) = n$.

Let $\mathfrak{R} \in \mathcal{H}(X)$, with $D_E(\mathfrak{R}) = n$ and $D_T(\mathfrak{R}) = k \leq n$.

Let $\mathfrak{R}_i^1 \in \mathcal{H}(X)$, such that $\mathfrak{R}_i^1 \subset \mathfrak{R}$ for $i = 1, 2, \cdots m$

Then, $\mathfrak{R}$ is a *regular fractal* if there exists an IFS of homeomorphisms $\{\mathbf{f}_i\}$, such that:

$$\mathbf{f}_i \colon \mathfrak{R} \to \mathfrak{R}_i^1 \qquad \forall i \in \{1, 2, \cdots m\}$$

And

$$\bigcup_{i=1}^{m} \mathbf{f}_i(\mathfrak{R}) = \mathfrak{R}$$

Inductively, every $\mathfrak{R}_i^1$ is a regular fractal as well [4]. And that sets the first notion of invariance that will help us define what a fractal dimension is, so first we should begin defining a dimension for regular fractals.

Before anything can be said for the definition of fractal dimensions a remark should be made, and it should be clear that several different approaches can be taken to define it. Usually taking things from probability or measure theory in order to define how precise the dimensions should be. In this instance we will take a little bit of measure theory in order to have a more direct approach according to the material that we already have.

Enter the domain of fractal dimensions, we begin with the first definition of dimension, *similarity dimension* (D_S), we should recall that the space we will be working from now is $(\mathcal{H}(X), h)$ and the definition of regular fractal still holds for it.

[4] Since we can find a subset $\mathfrak{R}_{i'}^2 \subset \mathfrak{R}_i^1$, such that $\mathbf{f}_{i'}\left(\mathfrak{R}_i^1\right) = \mathfrak{R}_{i'}^2 \to \mathbf{f}_{i'}\left(\mathbf{f}_i\left(\mathfrak{R}\right)\right) = \mathfrak{R}_{i'}^2$ and apply that inductively to get any number of iterations of $\mathbf{f}$.

Definition 1.2.4 (Similarity Dimension)

Let $\mathfrak{R}$ be a regular fractal on a complete metric space (X, d) in $\mathbb{R}^m$ with topological dimension k.

Also let $\{(\mathfrak{R}, d); \mathbf{f}_i : \mathfrak{R} \to \mathfrak{R}_i^1\}$ be the IFS of $\mathfrak{R}$, for $i = 1, 2, \cdots m$ with contractivity factors s_i for each $\mathbf{f}_i$.

Let r_i be the scaling factor of the Hausdorff metric in $\mathfrak{R}$ with respect to the one in $\mathfrak{R}_i^1$, such that the metric spaces are equivalent as follows:

$$h(\mathbf{f}_i(A), \mathbf{f}_i(B)) = r_i \cdot h(A, B) \tag{1.4}$$

For all $A, B \in \mathcal{H}(\mathfrak{R})$

We define *similarity dimension* as:

$$D_S(\mathfrak{R}) = \frac{\ln(P_i)}{\ln(r_i)} \qquad \forall i = \{1, 2, \cdots m\} \tag{1.5}$$

Where:

$$P_i = \frac{s_i}{\sum_{j=1}^{m} s_j}$$

And from that we get ${r_i}^{D_S} = P_i$. Where P_i is called the *partition factor of* $\mathfrak{R}_i^1$.

We will make it more clear with examples and explanations of each of them in the following section, but it is important to make clear that the scaling factor r_i of the metric is a “lengthçontraction and the contractivity factors are contractions of “area”, “volume.ºr its equivalent in $\mathbb{R}^n$.

1.3. Examples Of Regular Fractals

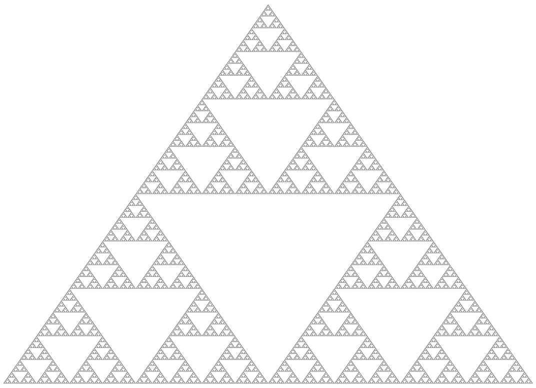

Figure 1.3: Regular Fractal: Sierpinski Triangle ($\mathfrak{s}$)

The first example we will show and explain how its fractal dimension is constructed is the Sierpinski triangle ($\mathfrak{s}$), figure 1.3. The main idea is to call the whole fractal $\mathfrak{R}$ and divide it into three equal parts $\{\mathfrak{R}_1^1, \mathfrak{R}_2^1, \mathfrak{R}_3^1\}$, an seen in figure 1.4, where its labeled and colored in red, blue and green, respectively.

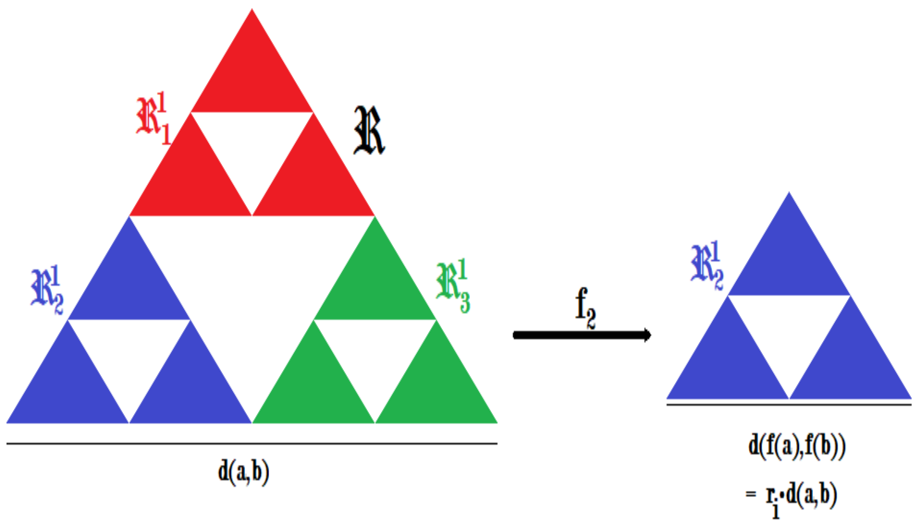

Figure 1.4: Construction of $\mathfrak{s}$ as a regular fractal

The contractivity factor of the component $\mathfrak{R}_2^1$ is $s_2 = \frac{1}{4}$, which gives the partition factor $P_2 = \frac{1}{3}$; and the corresponding metric scaling factor is $r_i = \frac{1}{2}$. This leads to the similarity dimension of $\mathfrak{s}$, which is $D_S(\mathfrak{s}) = \dfrac{\ln(1/3)}{\ln(1/2)} \simeq 1{,}585$.

The same reasoning and processes can be applied to any regular fractal, take for example the Menger carpet ($\mathfrak{m}_2$), seen in figure 1.5, which consists of 8 equal parts whose metric scaling factor is 1/3, and partition factor of 1/8. Giving us the similarity dimension of $D_S(\mathfrak{m}_2) = \dfrac{\ln(1/8)}{\ln(1/3)} \simeq 1{,}893$.

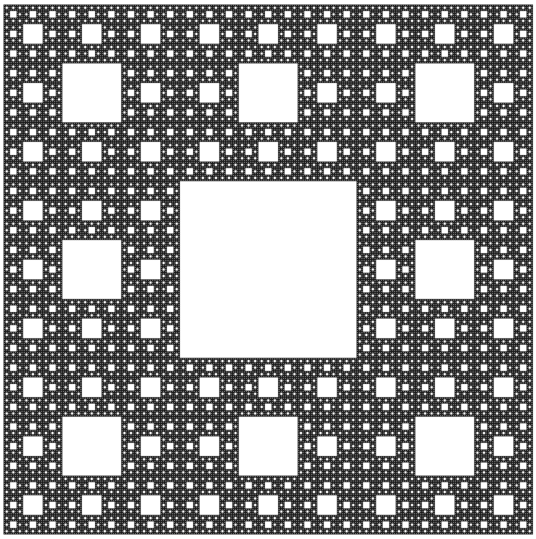

Figure 1.5: Regular Fractal: Menger Carpet

To finish this chapter some regular fractals along their fractal dimension will be presented.

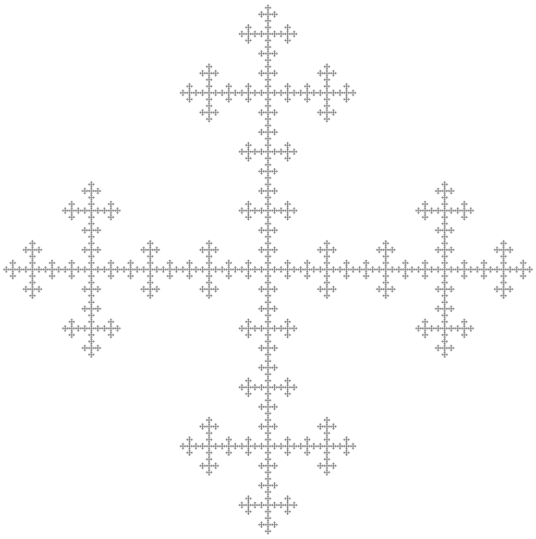

Figure 1.6: Regular Fractal: $D_S = \dfrac{\ln(1/5)}{\ln(1/3)} \simeq 1{,}465$

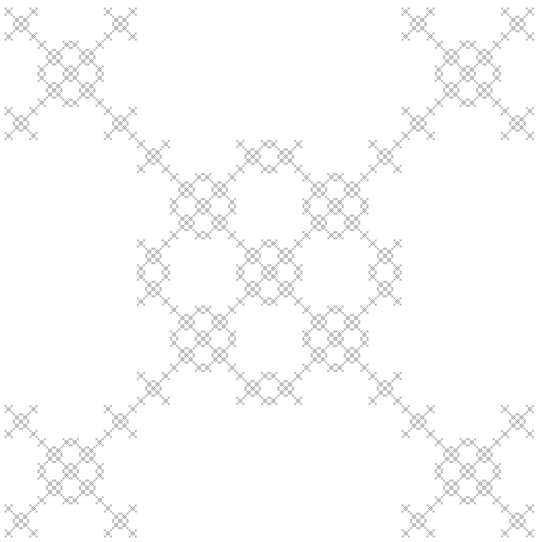

Figure 1.7: Regular Fractal: $D_S = \dfrac{\ln(1/8)}{\ln(1/4)} = 1{,}5$

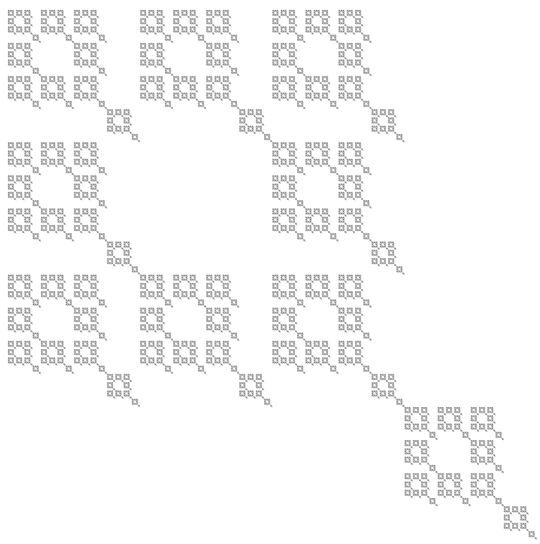

Figure 1.8: Regular Fractal: $D_S = \dfrac{\ln(1/10)}{\ln(1/4)} \simeq 1{,}661$

Measure on Fractals

An important idea to introduce within the ideas of fractal dimension is the usage of a measure for fractals. That will lead to a proper development of the “mass.ºr “density.ºf a subspace $X \subseteq \mathbb{R}^n$. Since the ideas behind measure theory are not in the scope of this document, we will let the construction of the Lebesgue Measure for the Analysis books [5] and we will jump straightforward to the definition of *measure*, specifically, the *Lebesgue Measure* ($\mathcal{L}$).

Definition 1.3.1 (Lebesgue Measure)

Let $\{O_\alpha | \alpha \in \Delta, \Delta$ as a family of indexes$\}$ be a collection of pairwise disjoint finite open cells in $\mathbb{R}^n$. We define the *Lebesgue Measure of* O_α as a real number l_α:

$$\mathcal{L}(O_\alpha) = \left[\prod_{k=1}^{n}(b_k - a_k)\right]_\alpha = l_\alpha \tag{1.6}$$

Let O be an open set defined by the union of all sets in $\{O_\alpha\}$ to define the *Lebesgue Measure of* O as follows:

$$\begin{aligned}\mathcal{L}(O) = \mathcal{L}\left(\bigcup_{\alpha\in\Delta} O_\alpha\right) &= \sum_{\alpha\in\Delta}\mathcal{L}(O_\alpha) \\ = \sum_{\alpha\in\Delta}\left[\prod_{k=1}^{n}(b_k - a_k)\right]_\alpha &= \sum_{\alpha\in\Delta} l_\alpha\end{aligned} \tag{1.7}$$

Since it is a well known fact, that some sets have Lebesgue measure 0 in an Euclidean space, like the cantor set ($\mathfrak{C}$) in $\mathbb{R}$. Other kind of measure should be implemented in those fractals to develop some intuition on how fractals work, so we can begin by refining the space we are currently working with.

This leads to a question on how to define a measure on these sets. A common one is to define one through the IFS. And it makes sense if we think of any fractal as measure 1, then the w_i IFS should have a corresponding measure of P_i. We can consider this as a property of any fractal, but we wont delve more into this, since we are more interested in the topological properties of the fractals. For more information, Barnsley gives a probability and measure theory introduction on fractals on his book [1].

Chapter 2

Self-Similarity

As stated in the introduction, a fractal is a complete metric space that is self-similar at any scale. In this section we will broadly define what self-similarity is, by introducing a couple of terms.

Box Counting

The same way the Lebesgue Measure is constructed, we can add a new type of measure on a compact set. It is important to remark that a better way to do it is through probability, but another path can be taken in order to get the desired result. So first we should the proper environment for us to work.

Definition 2.0.1 (Box Count)

Let (X, d) be a complete metric space, and let $A \in \mathcal{H}(X)$ and $D_E(A) = n$.

Let $C^n_\delta(A)$ be the minimum closed n-cell of side lengths $\delta > 0$, such that $A \subseteq C^n_\delta(A)$.[a] For simplicity, we will just denote it as C^n.

Let $\mathcal{C}_k$ be the refinement of $\{C^n\}$ that contains 2^k closed n-cells of side length $\delta/2^k$, call it k-grid of C^n. Such that:

$$C = \bigcup_{E \in \mathcal{C}_k} E$$

We define the *Box count of* A *in* $\mathcal{C}_k$, $N_k > 0$, as the number of boxes in $\mathcal{C}_k$ that follow the condition: $E \in \mathcal{C}_k$ has $p \in A$ for some $p \in E$.

[a] $C^n_\delta(A)$ is a cover of A.

With this we can see that the box count of a compact set A depends on how the cell is divided[1]. And continuously taking half subdivisions of a cell, refines the box count. With this, we define:

Definition 2.0.2 (Box-Counting Pre-Dimension)

Let (X, d) be a complete metric space, and let $A \in \mathcal{H}(X)$.

Let $N_k(A)$ be the box count of A in $\mathcal{C}_k$, then the k *Box-counting dimension*, $D_{B,k}$, is defined as:

$$D_{B,k}(A) = \frac{\ln(N_k(A))}{\ln(2^k/\delta)} \tag{2.1}$$

An important thing to remark here is that the number $D_{B,k}(A)$ is bounded by the euclidean dimension $D_E(A) = n$. considering that for an arbitrary $A \in \mathcal{H}(X)$ the maximum number of boxes we can get is $\text{máx}\{N_k(A)\} = N_k(C^n) = 2^{nk}$, for

[1] Note that the order of a cover is not defined on closed covers.

$\delta = 1$, we get:

$$D_{B,k}(C^n) = \frac{\ln(N_k(C^n))}{\ln(2^k)} = \frac{nk\ln 2}{k\ln 2} = n \tag{2.2}$$

Hence,

$$D_{B,k}(A) \leq D_E(A) = n \tag{2.3}$$

This fact will be really useful in the further chapter about self-similarity.

For a more visual representation, a picture such as the one in figure 2.1 may serve as a guide on how the pre-dimension works.[2]

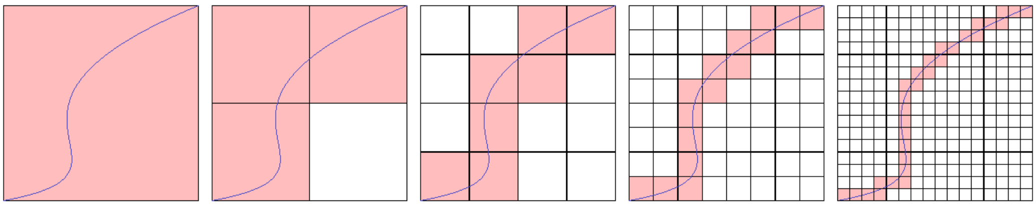

Figure 2.1: Box count of a curve A in a cell of side-length δ.

	Side Length	n-cells Area	$N_k(A)$	$D_{B,k}(A)$
1	δ	δ	1	N.D.
2	$\delta/2$	$\delta/4$	3	1,585
3	$\delta/4$	$\delta/16$	7	1,404
4	$\delta/8$	$\delta/64$	15	1,302
5	$\delta/16$	$\delta/256$	30	1,227

Table 2.1: Box Count and pre-dimension for figure 2.1

The box-counting dimension may not be seen as a fractal dimension, for reasons that will be explained later, but we will call it a pre-dimension since it helps constructing with a sequence of compact sets one of the most commonly used fractal dimensions, the *Minkowski–Bouligand dimension*, D_{MB}:

[2]Considering we take $\delta = 1$

Definition 2.0.3 (Minkowski–Bouligand Dimension)

Let $A \in \mathcal{H}(X)$.

The *Minkowski–Bouligand dimension of A* is:

$$D_{MB}(A) = \lim_{k \to \infty} [D_{B,k}(A)] \tag{2.4}$$

Since the Minkowski–Bouligand dimension of a set A can be found by finding the limit of a sequence of integer numbers determined by the sequence of box-counting dimensions. The proof of the convergence of this sequence can follow from the idea that we can take the cover $\mathcal{C}_k$ as a bounded sequence of compact sets and the fact that if C^n is the δ sided n-cell that contains A, then the n-volume of the n-cell is greater than the product of the counted boxes times the n-volume of said boxes, i.e.

$$\delta^n = \left(\frac{\delta}{2^k}\right)^n 2^{nk} = \left(\frac{\delta}{2^k}\right)^{D_E(A)} 2^{nk} \geq \left(\frac{\delta}{2^k}\right)^{D_{B,k}(A)} N_k(A)$$

This leads to the convergence of said sequence, but for further explanations Barnsley gives a detailed explanation of the proof. [1]

2.1. Clusters

Before introducing a proper measure on fractals, two ideas should be introduced to our collection of terms. The first is pretty straightforward and begins with the idea of a *closed cover*. The same way an open cover on a set X is defined with open sets that contain a subset, a closed cover is a collection of the closure of open sets, which are closed sets, and those sets contain the set X. This allows us to define:

Definition 2.1.1 (Closed Cluster)

Let $\mathcal{B} = \{C_\alpha\}$ be a closed cover of a space (X, d). If every C_α is a closed ball $\overline{B}(x)$ with $x \in X$ we call the closed cover $\mathcal{B}$ a *closed cluster*.

Furthermore, if every closed ball in the cover is a closed ball of radius ϵ, $\overline{B}_\epsilon(x)$, we denote the cluster as $\mathcal{B}_\epsilon$ and call it *closed ϵ-cluster*.

With this, a cluster on a compact set should be well defined in a compact set, take for example the figure 2.2.

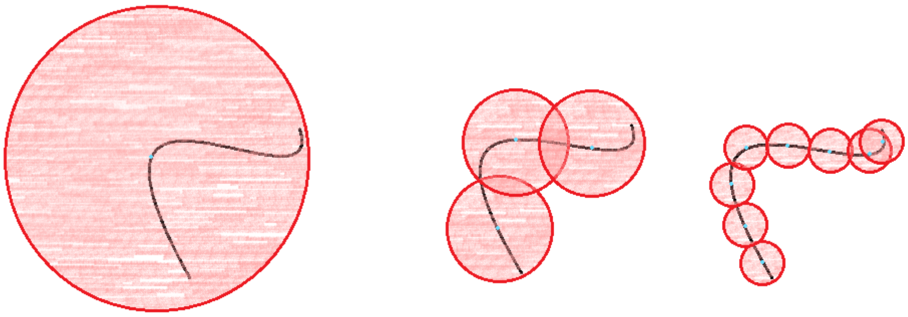

Figure 2.2: ϵ-clusters for a curve for different ϵ.

So, the idea behind this is to find how can we cover a set with closed balls of different radii in the case of a closed cluster and with balls of the same radius in the case of an ϵ-cluster. This idea is somewhat used in data science, so that information can be included in desired spaces.

With this the second idea can be introduced as a definition:

Definition 2.1.2

Let (X, d) be a complete metric space . Let $A \in \mathcal{H}(X)$.

We denote the *minimum* ϵ*-cluster number of* A, $\mathcal{N}_\epsilon(A)$, as follows:

$$\mathcal{N}_\epsilon(A) = \text{ínf}\left\{ M \middle| A \subset \bigcup_{i=1}^{M} \overline{B}_\epsilon(x_i)\ ,\ x_i \in A\ ,\ \overline{B}_\epsilon(x_i) \in \mathcal{B}_\epsilon \right\} \tag{2.5}$$

For some $\mathcal{B}_\epsilon$ closed ϵ-cluster of A.

That is, the minimum number of closed balls of radius ϵ that contain the compact set A.

Hausdorff Dimension

Once the box-counting dimension and minimum ϵ-cluster definitions are introduced, we can begin defining the Hausdorff Dimension, which is the most widely and used fractal dimension, being called in some text books ”the fractal dimension.ºf a compact set [1].

At a first glance, the idea of box counting seems related with the idea of the minimum ϵ-cluster. But there are some topological considerations that the former takes and the second oversees and vice versa. For example, the box count is defined recursively, and the idea of finding a minimum ϵ-cluster number considers the infimum cover of a collection of sets. With that in mind, we define:

Definition 2.1.3 (Hausdorff Dimension)

Let (X, d) be a complete metric space, and let $A \in \mathcal{H}(X)$.

Then for each $\epsilon > 0$, there exists a minimum ϵ-cluster number of A, $\mathcal{N}_\epsilon(A)$, then we define:

$$D_{HB}(A) = \lim_{\epsilon \to 0} \left\{ \frac{\ln \mathcal{N}_\epsilon(A)}{\ln(1/\epsilon)} \right\} \tag{2.6}$$

And it is called the *Hausdorff-Besicovitch* or *Hausdorff Dimension of A*.

The idea of Hausdorff dimension follows from the fact that the smaller the radius of the balls in the minimum ϵ-cluster, the more balls with radius ϵ are needed. Furthermore, if we allow the quotient D_{HB}, such that as $\epsilon \to 0$, we have:

$$\mathcal{N}_\epsilon(A)\epsilon^{D_H} \to 1 \tag{2.7}$$

That coefficient is in fact the Hausdorff dimension. This will finally lead us to our final proposition for our study on fractals.

Proposition 3

Let (X, d) be a complete metric space. Then, for any Fractal $\mathfrak{F} \in \mathcal{H}(X)$ we have:

$$D_{HB}(\mathfrak{F}) \leq D_{MB}(\mathfrak{F}) \leq D_E(\mathfrak{F}) \tag{2.8}$$

Proof To prove this proposition, we will prove first why $D_{MB} \leq D_E$, followed immediately with $D_{HB} \leq D_{MB}$; the proof of each part will be denoted by a white square, $\square$.

As it was stated before, the construction of the Minkowsky-Bouligand dimension is recursive, this will show that for a set $A \in \mathcal{H}(X)$ and a δ sided cell, we check the recursively defined dimension and its boundaries:

$$\left(\frac{\delta}{2^k}\right)^{D_{B,k}(A)} N_k(A) \leq \left(\frac{\delta}{2^k}\right)^{D_E(A)} (2^k)^{D_E} = \delta^{D_E(A)} \tag{2.9}$$

For all $k > 0$, then by taking the limit of that bounded sequence $D_{B,k}(A)$, we get $D_{MB} \leq D_E$. $\square$

The second inequality is proven by comparing the covers of the box count and the minimum ϵ-cluster number of A as follows, so, without loss of generality, let the length of the n-cell be $\delta = 1$:

Let the minimum $(1/2)^k$-cluster number of A be $\mathcal{N}_{(1/2)^k}(A)$, and the box count of A in $\mathcal{C}_k$ be $N_k(A)$. Then.

$$\mathcal{N}_{(1/2)^k}(A)\frac{1}{2^k} \leq N_k(A)\frac{1}{2^k} \tag{2.10}$$

Since not only every ball of radius $1/2^k$ contains every cell of side length $1/2^k$, but also $\mathcal{N}_k(A)$ considers the minimum balls needed to cover. Hence, it is at most, $N_k(A)$. Therefore as $k \to \infty$ we get $D_{HB} \leq D_{MB}$.$\square$

This completes the proof. $\blacksquare$

Now, we have established a chain of different dimensions for fractals that are determined by the coverings of the fractal. The last thing to prove is that the topological dimension is less than or equal to the fractal dimensions. If suffices to prove $D_T(\mathfrak{F}) \leq D_{HB}(\mathfrak{F})$, that way we can finish with one of the major consequences of fractal geometry, which is called the Sznirelman theorem.

Theorem 2.1.1 (Sznirelman Theorem)

Let (X, d) be a complete metric space.

Then, for any compact set $A \in \mathcal{H}(X)$ we have:

$$D_T(A) \leq D_{HB}(A) \tag{2.11}$$

The demonstration of this theorem goes far beyond the scope of the areas studied on this parer, since deep understanding on dimension theory, stochastic analysis and measure theory are needed. So, as a reference, *Dimension Theory* by W. Hurewicz and H. Wallman will serve as a guide [2], specifically chapter VII, section 4. A short statement is said about this as a Corollary on the study of dimension theory.

But to avoid the reader from leaving empty handed, the basic idea is to prove that the measure of the Hausdorff cover is at least the measure of the minimum measure of the minimum refinement of a set, which determines the topological dimension. This leads to $D_T(A) \leq D_{HB}(A)$.

With everything said so far, we can say that fractal dimensions are ratios that describe the topological or statistical complexity of a compact set. So, far we have just seen three types of fractal dimensions: D_S, D_{MB} and D_{HB}; $D_{B,k}$ is not a fractal dimension in the same sense since it can vary according to the index k and may describe some global properties of the compact set that may as well not be true locally or vice versa. So hesitantly and for the dimensions we have, we define:

Definition 2.1.4 (Self-Similarity)

A set $A \in \mathcal{H}(X)$ is *self similar* if it preserves any fractal dimension under an affine transformation.

One last Example

The unfolding dragon, $\mathfrak{D}$, in figure 2.3 is a surface in $\mathbb{R}^2$ with a fractal boundary, even though it looks like a regular fractal, it doesn't follow the definition we have for regular fractal. But despite that, the boundary of it is self similar, hence it is a fractal.

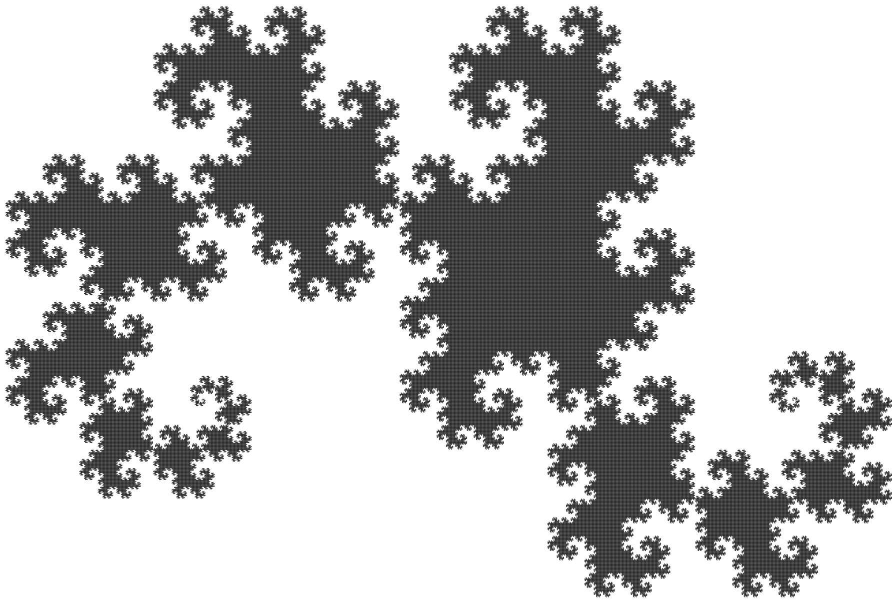

Figure 2.3: Unfolding Dragon

Chapter 3

Further Reading and Conclusions

Our definition of fractal is still a loose definition, since self similarity has been introduced very broadly. But with the knowledge that we have about fractals right now we can say that self-similarity implies the conservation of any fractal dimension. That means, the fractal dimension of a set is the same for most subsets of the fractals. More fractal dimensions can be defined, but that would go way beyond the scope desired for this paper, nevertheless some will be listed below as sources of investigation for further readings.

- Homothety Dimension (Generalization of Similarity Dimension)
- Information Dimension
- Correlation Dimension
- Rényi Dimension
- Packing dimension (Coarser variant of Hausdorff dimension)

Other Areas that will help a lot in the study of fractals are Measure Theory, to determine how different measures help us understand fractals and their measures; Dynamical Systems, even though it was not mentioned, through attractors many fractals can be uniquely defined; Dimension Theory, as already mentioned before, several properties of fractals follow from the study of dimension theory; Algebraic topology, it was not a topic discussed in here, but algebraic topology can describe fractals through homology, cohomology and homotopy families. Also Ring and

Group theory can be used to show other interesting properties, for example in figure 3.1 you can see that some fractals can be used as tiles of a space.

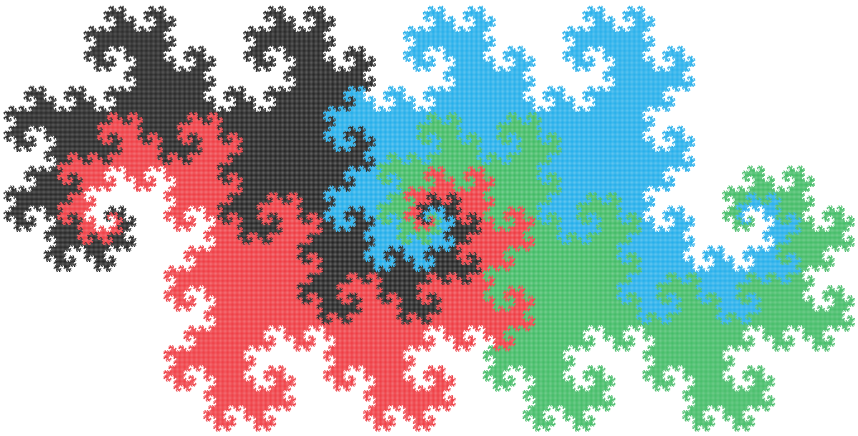

Figure 3.1: Unfolding Dragon Tiling

The study of fractal topology merely begins from here. Since there are several areas that can be explored in order to have a new understanding of nature. Mandelbrot already said that nature has fractal geometry. And it was not far from the truth. Trying to describe mountains with cones, planets with spheres and everything with regular shapes will work up to a point, and refinements may not only be tedious, but won't give a proper description of the space where we are working in.

Fractal geometry and topology may not give all the answers we need, but studying it will lead to several breakthroughs since it is related to so many areas and its our responsibility as conscious being to try to delve as much as we can into the the unknown areas of math and science. And what better area to develop than the study of nature's geometry itself?

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